THE RELATIVE CANONICAL ALGEBRA FOR GENUS 3 FIBRATIONS

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Summary

This thesis studies surfaces with fibrations by curves of small genus, usually locally in a neighbourhood of a singular fibre. The main method is algebraic, consisting of describing the relative canonical algebra. In the genus 2 case there are classical results of Horikawa that have been used successfully by Xiao Gang to get global results on surfaces. This thesis is mainly concerned with the genus 3 case, which is much harder; even here the results are not definitive. We prove that for genus 2 and 3 the relative canonical algebra is generated in degrees 1, 2, 3 and related in degree ≤ 6. In fact we give a much more detailed analysis of the ring by generators and relations.

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Introduction

This thesis studies surfaces with a fibration $f: S \to B$ of curves of genus 2 or 3, locally in a neighbourhood of a singular fibre.

For genus 2 fibrations, the hyperelliptic involution of a general fibre presents S as birational to a double covering of a ruled surface. In his now classical paper [Ho] , Horikawa studies this double cover, giving a classification of fibres in terms of the singularities of its branch locus. These results of Horikawa have been successfully used by Xiao Gang [XG] to obtain global results on surfaces with genus 2 pencils.

In the genus 3 case a geometrical approach generalizing Horikawa's would be quite complicated since it would involve studying birational problems of surfaces in ambient 3-folds.

Our main method is algebraic; it consists of describing the relative canonical algebra $\mathcal{R}(f, \omega_f)$ of the fibration, where $\mathcal{R}(f, \omega_f)$ is the graded $\mathcal{O}_B$-algebra $\bigoplus_{n\geq 0} f_* \omega_{S/B}^{\otimes n}$. This is essentially equivalent to studying the canonical ring $R(F,K_S)$ for the scheme-theoretic fibre.

Here we are mainly concerned with the genus 3 case but we also treat briefly the genus 2 case. We show that a genus 2 fibration is canonically a complete intersection in weighted projective space (see II.3.1). Locally over the base, it is either a weighted hypersurface of degree 6 in $\mathbb{P}_B(1,1,3)$ or a complete intersection in $\mathbb{P}_B(1,1,2,3)$ of a hypersurface of degree 2 and one of degree 6. Horikawa's analysis can be redone in these terms. The equation in degree 2 corresponds to the fibre not being 2-connected, and is of the form $X_1X_2=0$ or $X_1{}^2=0$. In the first case we obtain the fibres of type I_k and II_k of Horikawa's analysis and in the second case the fibres of type III_k, IV_k,V.

We show that the pluricanonical image of a 2-connected genus 3 fibre is also a complete intersection in weighted projective space, described either by one degree 4 equation in $\mathbb{P}^2$ or two equations in $\mathbb{P}(1,1,1,2)$, one of degree 2, one of degree 4. This distinction corresponds to the fibre being hyperelliptic or not.

However, this simple description is lost as soon as we drop the assumption of 2-connectedness, and the canonical rings turn out to need a large number of generators and relations.

In this thesis we prove that these canonical rings are generated by their elements of degree ≤ 3 (II.5.5) and related in degree ≤ 6. To obtain this result a more explicit analysis is required. We show that the rings of 1-connected fibres can naturally be divided in 4 different cases, according to the decomposition of the fibres; the number of generators (III.1.18) depends on the case. For reduced fibres these cases correspond to the number of 2-connected divisors E with $E^2=-1$, $KE=1$ (elliptic tails) contained in the fibre. The canonical rings are described explicitly by generators and relations in Chapter III.

Methods: the methods used to describe $R(F,K_F)$ are mainly classical, such as Castelnuovo's lemma. Some of our methods can be regarded as an adaptation for reducible curves of the Petri analysis.

In order to study $R(D, L)$, where D is a reducible divisor on a surface and L an invertible sheaf on D, a good approach is to try to find a suitable decomposition of D as a sum of two divisors A and B. To obtain such a decomposition we exploit the notion of m-connectedness of D. If the two divisors A and B do not have common components (notice that this does not imply that D is reduced) then $R(D, L)$ can also be obtained as a subring of $R(A, L) \oplus R(B, L)$. If A and B have common components then one has to study the kernel of the restriction maps and try to find out which of the relations on A and B lift to $R(D, L)$, and how. In fact in this thesis we use mainly this second approach, since it allows us to give a unified treatment of the reduced and non-reduced cases.

The canonical ring of a 2-connected fibre for both genus 2 and 3 cases is quite easy to study: it has essentially the same description as the ring of an irreducible fibre.

For genus 2 and 3 fibres, a non-multiple fibre can only fail to be 2-connected if it contains a 2-connected divisor E with $E^2=-1$, $KE=1$. In this case the restriction map from $R(F, K_F)$ to $R(F-E, K_F)$ is surjective and it turns out that its kernel is a principal ideal generated by an element of degree 1. Using this fact, by a standard procedure (II.1), we recover $R(F, K_F)$ once $R(F-E, K_F)$ is known. The calculation here is very similar to deformation calculations. An easy example of this method is the calculation in (II.3).

Open questions I am convinced that essentially the same methods used to prove that the canonical algebra is generated in degree ≤ 3 for genus 3 fibres can be used to prove the same statement for any genus. The proof we give is perhaps related to Laufer's proof of a similar statement for surfaces singularities [L].

It would be interesting to see how this rough classification of genus 3 fibres can be used to find numerical invariants for global surfaces with genus 3 fibrations. Xiao Gang conjectured that a neighbourhood of a "bad" fibre should have a Morsification, that is a small deformation where each singular fibre either contains a single node or is a non-singular multiple fibre. This would follow if we knew that the deformation theory of the fibre is unobstructed.

CHAPTER I

Preliminary results on divisors on surfaces.

Section 1. Basic facts and terminology.

Notation Surface means smooth projective surface over $k = \mathbb{C}$.

We denote by $k[X,\ldots]$ (capital letters) the polynomial ring in the weighted variables $X, \ldots$. In most cases we will denote by X_i variables of weight 1, Y_j variables of weight 2 and Z_k variables of weight 3.

Let D be an effective divisor on a surface S. Since D is Gorenstein the dualizing sheaf ω_D is invertible on D and we have the adjunction formula

(1.1) $$\omega_D = \omega_S \otimes \mathcal{O}_S(D) \otimes \mathcal{O}_D .$$

Throughout we will denote ω_D by K_D and thus the adjunction formula can be written

(1.2) $$K_D = (K_S + D)|_D .$$

By definition the *arithmetical genus* $p_a(D)$ of D is

(1.3) $$p_a(D) = 1 - \chi(\mathcal{O}_D) = h^1(D,\mathcal{O}_D) - (h^0(D,\mathcal{O}_D) - 1)$$

and by adjunction

(1.4) $$2\,p_a(D) - 2 = D(K_S + D) .$$

The usual duality theorems for smooth curves also hold : if L is a locally free sheaf of finite rank on D then

(1.5) $$H^0(D, L) \cong H^1(D, K_D \otimes L^\vee)^\vee$$
$$H^1(D, L) \cong H^0(D, K_D \otimes L^\vee)^\vee \qquad \text{where } \vee \text{ means dual} .$$

We also have:

(1.6) **Theorem** (Riemann-Roch for an effective divisor on a surface). Let D be an effective divisor on a surface and F a locally free sheaf of rank r on D. Then

$$\chi(F) = \deg(F) + r\,\chi(\mathcal{O}_D).$$

(1.7) If $D = A + B$ (with $A > 0$, $B > 0$) we have the exact sequence (see [B-P-V] II.1).

$$0 \longrightarrow \mathcal{O}_A(-B) \longrightarrow \mathcal{O}_D \xrightarrow{r} \mathcal{O}_B \longrightarrow 0$$

which will be called a *decomposition sequence* for $D = A + B$.

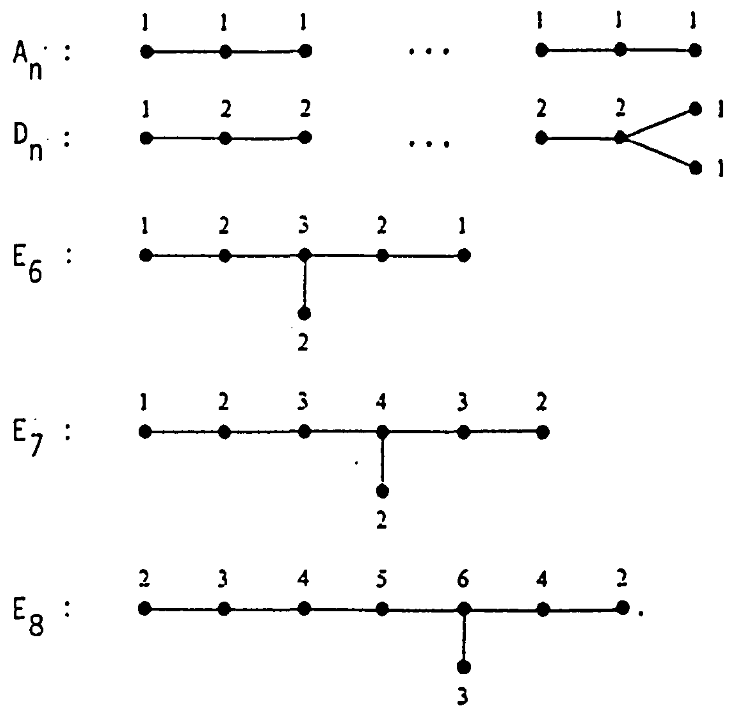
(1.8) **Notation** If L is an invertible sheaf on a divisor D we will denote $L^{\otimes n}$ by nL and the graded ring $R(D, L)$ is defined by

$$R(D, L) = \bigoplus_{n\geq 0} H^0(D, nL).$$

The multiplication in $R(D, L)$ is the usual product derived from $L^{\otimes n} \otimes L^{\otimes m} \to L^{\otimes n+m}$.

(1.9) **Definition** A (-2)-*curve* is a curve θ such that $\theta \cong \mathbb{P}^1$ and $\theta^2 = -2$. We will say that a divisor D is a (-2)-*cycle* if $D^2 = -2$ and every component Γ of D is a (-2)-curve such that $\Gamma D \leq 0$.

It is well known that the only possible types of (-2)-cycles are represented by the following Dynkin diagrams (where the numbers represent the multiplicity of each curve in D)



A (-2)-cycle D is of type A_n if and only if D is either irreducible or D contains two (-2)-curves θ_1 , θ_2 such that $\theta_1 \neq \theta_2$ and $\theta_1 D = \theta_2 D = -1$. If D is not of type A_n then D contains a unique (-2)-curve θ , appearing with multiplicity 2 in D , such that $\theta D = -1$.

(1.10) Definition A divisor D (not necessarily effective) on a surface S is *nef* (numerically effective) if $DC \geq 0$, for every curve C in S .

(1.11) Definition Two divisors in D , D′ (not necessarily effective) on a surface are *numerically equivalent* if DE = DE′ , for every E divisor in S .

For all the facts in this section refer to [B-P-V].

Section 2. m-connectedness.

(2.1) Definition For $m \geq 0$, an effective divisor on a surface is (numerically) m-connected if for every decomposition $D = D_1 + D_2$ (with $D_1, D_2 > 0$) $D_1 D_2 \geq m$.

The following easy lemmas will be used repeatedly in the sequel.

(2.2) Lemma Let D be a divisor which is m-connected and $D = D_1 + D_2$ ($D_1, D_2 > 0$) be a decomposition such that $D_1 D_2 = m$. Then (with $[x]$ denoting the integer part)

(i) D_1 and D_2 are $[\frac{m+1}{2}]$-connected.

(ii) If D_1 is chosen to be minimal subject to the condition $D_1(D-D_1) = m$ then D_1 is $[\frac{m+3}{2}]$-connected.

Proof (i) Let $D_1 = A_1 + A_2$ (with $A_1, A_2 > 0$). Then $A_1(A_2 + D_2) \geq m$ and $A_2(A_1 + D_2) \geq m$, by m-connectedness of D. Since $(A_1 + A_2)D_2 = D_1 D_2 = m$, it follows that

$$2A_1A_2 \geq m, \text{ that is } A_1A_2 \geq \frac{m}{2};$$

(ii) By the assumption on D_1, $A_1(A_2+D_2) \geq m+1$, so the proof of (ii) is the same as (i) with m replaced by $m+1$.

(2.3) Lemma If a divisor D is m-connected but not $(m+1)$-connected and $D_1 \subset D$ (with $D_1 > 0$) is minimal subject to $D_1(D-D_1) = m$ then either:

(i) $D_1 \subset D-D_1$ or

(ii) D_1 and $D-D_1$ have no common components.

Proof Let $D_1 = A + B$, $D - D_1 = A + C$ where B,C have no common components; then $BC \geq 0$ and so

$$m = D_1(D-D_1) = A^2+AB+AC+BC \geq A^2+AB+AC = A(D-A).$$

Thus by minimality of D_1 with respect to $D_1(D-D_1) = m$ either $A = 0$ or $A = D_1$.

Section 3. Properties of invertible sheaves on a divisor.

(3.1) Lemma (see [B-C]). Let D be an effective divisor on a surface, L an invertible sheaf on D and $s \in H^0(D, L)$ with $s \neq 0$. Then either

(i) s does not vanish identically on any component Γ of D and $\deg_{D'} L \geq 0$, for every $D' < D$, hence $\deg_D L \geq 0$;

or

(ii) D is reducible, and if $D_1 < D$ is the biggest divisor such that $s_{|D_1} \equiv 0$, then $\deg_\Gamma (L-D_1) \geq 0$ for every $\Gamma \leq D-D_1$. In particular $D_1(D-D_1) \leq \deg_{D-D_1} L$.

Proof If s does not vanish identically on any component of D the statement in (i) is clear. Otherwise let $D_1 < D$ be the maximal divisor such that s maps to zero under

$$H^0(D, \mathcal{L}) \to H^0(D_1, \mathcal{L}) .$$

Then $D-D_1$ and $s \in H^0(\mathcal{O}_{D-D_1}(\mathcal{L}-D_1))$ satisfy (i) so $\deg_\Gamma(\mathcal{L}-D_1) \geq 0$, for every $\Gamma < D-D_1$. In particular $D_1(D-D_1) \leq \deg \mathcal{L}_{|D-D_1}$.

(3.2) Corollary If D is a 1-connected divisor and $\mathcal{L}$ is an invertible sheaf on D such that $\deg_\Gamma \mathcal{L} \leq 0$ for every $\Gamma < D$, and $\deg_D \mathcal{L} < 0$, then $H^0(D, \mathcal{L}) = 0$.

(3.3) Corollary If D is a 1-connected divisor on a surface and $\mathcal{L}$ is an invertible sheaf on D such that $\deg_\Gamma \mathcal{L} \leq 0$, for every $\Gamma \subset D$ then either

(i) $h^0(D, \mathcal{L}) = 0$ or

(ii) $h^0(D, \mathcal{L}) = 1$ and $\mathcal{L} \cong \mathcal{O}_D$.

(3.4) Corollary Under the assumptions of (3.3), if $p_a(D) = 0$ and $\deg \mathcal{L}=0$ then $\mathcal{L} \cong \mathcal{O}_D$.

Proof By the Riemann Roch theorem we have $h^0(\mathcal{L}) - h^1(\mathcal{L}) = 1$ and so in particular $h^0(\mathcal{L}) \neq 0$.

(3.5) Lemma Let D be a 1-connected divisor on a surface and P a non-singular point of D. If $h^0(D, \mathcal{O}_D(P)) > 1$ then

(i) The unique component Γ of D containing P is isomorphic to $\mathbb{P}^1$.

(ii) If D is reducible there exists D' such that $\Gamma < D' < D$ and $D'(D-D') = 1$.

(iii) If $P' \in \Gamma$ is any other non-singular point of D, then also $h^0(D, \mathcal{O}_D(P')) > 1$.

Proof (i) Let Γ be the only component of D containing P. Then $h^0(\Gamma, \mathcal{O}_\Gamma(P)) \geq 2$, hence $\Gamma \cong \mathbb{P}^1$.

(ii) Furthermore if D is reducible the sheaf $\mathcal{O}_D(P)$ has degree 1 on Γ and degree 0 everywhere else. Since $h^0(D, \mathcal{O}_D(P)) > 1$ there exists $s \in H^0(D, \mathcal{O}_D(P))$ vanishing on some $D' < D$, $D' \neq 0$. By Lemma (3.1), if we take D' maximal with that property then

$$D'(D-D') \leq \deg \mathcal{O}_D(P)|_{D-D'} = 1 .$$

This proves (ii).

(iii) follows from (i) and (ii).

(3.6) Lemma (Castelnuovo's lemma) Let D be a 1-dimensional projective scheme over an algebraically closed field k and L an invertible sheaf on D generated by its global sections. If F is a coherent sheaf on D such that $H^1(F \otimes L^{-1}) = 0$ then the map $H^0(F) \otimes H^0(L) \to H^0(F \otimes L)$ is surjective.

Proof See [Mu].

(3.7) Lemma (Free pencil trick). Let D be an effective divisor on a surface and L an invertible sheaf on D. Let s_0 be a global section of L not vanishing identically on any component of D and s_1 a global section of L having no common zeros with s_0. If M is an invertible sheaf on D then the kernel of the map

$$\varphi : H^0(M) \oplus H^0(M) \longrightarrow H^0(M \otimes L)$$

$$(\alpha, \beta) \longrightarrow \alpha s_0 + \beta s_1$$

is isomorphic to $H^0(\mathcal{M} \otimes L^{-1})$.

Proof $\quad$ $\operatorname{Ker} \varphi = \{(\alpha, \beta) \in H^0(\mathcal{M}) \oplus H^0(\mathcal{M}) : s_0\alpha = -s_1\beta\}$.

If $s_0\alpha = -s_1\beta$ then in particular β vanishes at the points P such that $s_0(P) = 0$. Since s_0 generates L everywhere else we can write $\beta = s_0 r$ where $r \in H^0(\mathcal{M} \otimes L^{-1})$ and so $s_0\alpha = -s_0 s_1 r$. By hypothesis s_0 is regular and thus $\alpha = -s_1 r$.

So $\operatorname{Ker} \varphi = \{(s_0 r, s_1 r) : r \in H^0(\mathcal{M} \otimes L^{-1})\} \cong H^0(\mathcal{M} \otimes L^{-1})$.

Section 4. Some lemmas.

We will need the following facts:

(4.1) **Definition** ([B], p.178). Let D be an effective divisor on a surface S; define $\alpha(D) = \dim \ker (H^1(S,\mathcal{O}_S) \to H^1(D,\mathcal{O}_D))$.

(4.2) **Lemma** If D is an effective divisor on a surface S then

$$\alpha(D) = h^1(S,\mathcal{O}_S(-D)) - h^0(D,\mathcal{O}_D) + 1 .$$

Proof The result follows immediately from the cohomology sequence of the exact sequence of sheaves

$$0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0 .$$

(4.3) **Lemma** (Francia) Let D be an effective divisor on a surface S. If $\sigma: S \to S'$ is a birational map between smooth surfaces then $\alpha(\sigma_* D) = \alpha(D)$.

Remark This lemma is due to P. Francia. For lack of a suitable reference we include a sketch of the proof.

Sketch proof

Step 1. For a smooth surface X over $\mathbb{C}$, $\mathrm{Pic}^0(X)$ is an Abelian variety and for a divisor C on X

$$\alpha(C) = \dim(\text{Zariski tangent space at } 0 \text{ of } \ker\{\mathrm{Pic}^0(X) \to \mathrm{Pic}\, C\}).$$

Step 2. $\mathrm{Pic}^0(S) = \mathrm{Pic}^0(S')$ because σ is a birational map. Since σ is a sequence of blow-ups it is not hard to prove that $\ker(\sigma^*: \mathrm{Pic}(\sigma_* D) \to \mathrm{Pic}\, D)$ is an algebraic group not containing any Abelian variety as a factor.

Step 3. Consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}^0(S) & \xrightarrow{r} & \mathrm{Pic}\, D \\ \| \sigma^* & & \uparrow \sigma^* \\ \mathrm{Pic}^0(S') & \xrightarrow{r'} & \mathrm{Pic}(\sigma_*(D)) \end{array} .$$

Let A_1, A_2 be the connected components at 0 of $\ker r$, $\ker r'$ respectively. Then A_2 is an Abelian variety and by Step 2 does not have any non-trivial morphism to $\ker(\sigma^*: \mathrm{Pic}(\sigma_* D) \to \mathrm{Pic}\, D)$. Hence $A_2 = A_1$ and the result follows from Step 1.

(4.4) Lemma Let $L = \mathcal{O}_S(L)$ be an invertible sheaf on a surface S. A point $P \in S$ is a base point of the linear system $|L|$ if and only if $h^1(\tilde{S}, \pi^*(L) \otimes I_E) > h^1(\tilde{S}, \pi^*(L))$ (where $\pi: \tilde{S} \to S$ is the blow-up at P and $E = \pi^{-1}(P)$ is the exceptional divisor).

Proof Let $P \in S$ and m_P its ideal sheaf. The sheaf $\mathcal{O}_S(L) \otimes m_P$ is the sheaf of germs of sections of L vanishing at P. Thus P is not a base point of $|L|$ if and only if

$$h^0(S, \mathcal{O}_S(L) \otimes m_P) \neq h^0(S, \mathcal{O}_S(L)).$$

Let $\pi : \tilde{S} \to S$ be the blow-up at P and $E = \pi^{-1}(P)$. Then $H^0(S, \mathcal{O}_S(L) \otimes m_P) \cong H^0(\tilde{S}, \pi^*(L) \otimes I_E)$ and the result follows immediately from the cohomology sequence of the exact sequence of sheaves

$$0 \to \mathcal{O}_{\tilde{S}}(\pi^*(L) \otimes I_E) \to \mathcal{O}_{\tilde{S}}(\pi^*(L)) \to \mathcal{O}_E \to 0.$$

(4.5) **Lemma** Let D be a 1-connected divisor on a surface S and P a multiple point of D. Let $\pi : \tilde{S} \to S$ be the blow-up of S at P and $E = \pi^{-1}(P)$. Then $\pi^*(D)-2E$ is 0-connected, and if $\pi^*D-2E = D'_1 + D'_2$ (with $D'_1, D'_2 > 0$) is a decomposition such that $D'_1 D'_2 = 0$ then $D'_1 = \pi^*D_1 - E$, $D'_2 = \pi^*D_2 - E$, where D_1, D_2 are nonzero divisors such that $D = D_1 + D_2$ (and thus $D_1 D_2 = 1$, $P \in D_1 \cap D_2$)

Proof The argument is well known (see for example [B-P-V], Ch. VII, §6).

(4.6) **Lemma** Let D be an effective divisor on a surface and $n \geq 0$ such that $H^0(S, \mathcal{O}_S(K+nD)) \neq 0$, $(K+nD)^2 > 0$ and K+nD is numerically effective. Then:

(i) Every $A \in |K+nD|$ is 1-connected.

(ii) If n is even then A is 2-connected.

Proof If A is irreducible there is nothing to prove. Otherwise $A = D_1 + D_2$, with $D_1, D_2 > 0$. Since $K+nD$ is nef, $D_1^2 + D_1D_2 \geq 0$ and $D_1D_2 + D_2^2 \geq 0$. As $(K+nD)^2 > 0$ at least one of these inequalities is strict, say, $D_1^2 + D_1D_2 > 0$.

By the index theorem $D_1^2 D_2^2 - (D_1D_2)^2 \leq 0$ and equality holds if and only if there exist $a,b \in \mathbb{Z}$ such that $aD_1 \approx bD_2$ (where $\approx$ denotes numerical equivalence).

If $D_1D_2 \leq 0$ we would then have

$D_1^2 > -D_1D_2 \geq 0$ and

$D_2^2 \geq -D_1D_2 \geq 0$,

and so $D_1^2D_2^2 > (D_1D_2)^2$, or $D_1^2D_2^2 = (D_1D_2)^2$ and $D_2^2 = 0$, contradicting in the first case the index theorem and in the second case $(K_S+nD)^2 > 0$. Thus $D_1D_2 \geq 1$ and so A is 1-connected. This proves (i).

Suppose that $D_1D_2 = 1$. Then

$$D_1^2 + 1 = D_1(D_1+D_2) = D_1(K+nD) = KD_1 + nD_1D .$$

Since n is even we would then have $KD_1 - D_1^2 \equiv 1 \bmod 2$, which contradicts the adjunction formula. Hence A is 2-connected.

Section 5. Base points of the linear system $|K_S + D|$.

(5.1) Theorem Let D be an effective divisor on a surface S and P a multiple point of D. Then P is a base point of the linear system $|K_S + D|$ if and only if

$$H^0(\mathcal{O}_{\pi^*D-E}) \to H^0(\mathcal{O}_{\pi^*D-2E}) \text{ is not surjective,}$$

where $\pi : \tilde{S} \to S$ is the blow-up of S at P and $E = \pi^{-1}(P)$.

Proof By (4.4) P is a base point of $|K_S + D|$ if and only if

$$h^1(\tilde{S},\pi^*(K_S + D) \otimes I_E) \neq h^1(S,\pi^*(K_S + D)) .$$

By Serre duality

$$h^1(\pi^*(K_S + D) \otimes I_E) = h^1(-(\pi^*D - 2E)) \text{ and}$$

$$h^1(\pi^*(K_S + D)) = h^1(-(\pi^*D-E)).$$

Since P is a multiple point on D both π^*D-E and π^*D-2E are effective divisors on $\tilde{S}$ and by (4.3),

$$\alpha(\pi^*D-2E) = \alpha(\pi_*(\pi^*D-2E)) = \alpha(D) \text{ and}$$

$$\alpha(\pi^*D-E) = \alpha(\pi_*(\pi^*D-E)) = \alpha(D).$$

Thus by (4.2) $h^1(-(\pi^*D-2E)) > h^1(-(\pi^*D-E))$ if and only if $h^0(\mathcal{O}_{\pi^*D-E}) < h^0(\mathcal{O}_{\pi^*D-2E})$, hence the result.

(5.2) Proposition Let D be a 1-connected divisor on a surface S and P a multiple point of D. Then P is a base point of the linear system $|K_S + D|$ if and only if D has a decomposition $D = D_1 + D_2$ such that

(i) $P \in D_1 \cap D_2$, $D_1D_2 = 1$

and

(ii) $h^0(\pi^*D_1-E, \mathcal{O}(-(\pi^*D_2-E)) \neq 0$

(with π, E as in (5.1)).

Proof Since D is 1-connected $h^0(\mathcal{O}_{\pi^*D-E}) = 1$ (by 3.3) and thus by (5.1) P is a base point of $|K_S + D|$ if and only if $h^0(\mathcal{O}_{\pi^*D-2E}) > 1$.

If $h^0(\mathcal{O}_{\pi^*D-2E}) > 1$, in particular (by 3.3) π^*D-2E is not 1-connected and then by (4.5) D has a decomposition $D = D_1 + D_2$ satisfying (i). Furthermore, by (2.2) both D_1 and D_2 are 1-connected and thus π^*D_2-E is 1-connected. Since $h^0(\pi^*D-2E) > 1$, (ii) follows from the cohomology sequence of

$$0 \to \mathcal{O}_{\pi^*D_1-E}(-(\pi^*D_2-E)) \to \mathcal{O}_{\pi^*D-2E} \to \mathcal{O}_{\pi^*D_2-E} \to 0 .$$

The converse is clear and so the result follows.

(5.3) Corollary 1 If D is a 2-connected divisor on a surface S, then no multiple point of D is a base point of the linear system $|K_S + D|$.

(5.4) Corollary 2 Let D be a reduced 1-connected divisor on a surface S. Then a multiple point P of D is a base point of $|K_S + D|$ if and only if $D = D_1 + D_2$ with $D_1 \cap D_2 = \{P\}$, $D_1D_2 = 1$ (i.e. P is a node of D and $D-\{P\}$ is not connected).

Section 6. The canonical sheaf K_D of a divisor D.

(6.1) Lemma Let D be a 1-connected divisor on a surface S. A non-singular point P of D is a common zero of the global sections of K_D if and only if $H^0(D,\mathcal{O}_D(P)) = 2$.

Proof Let P be a non-singular point of D. Then $m_P.\mathcal{O}_S(K_D)$ is an invertible sheaf. Considering the cohomology sequence of the exact sequence

$$0 \to m_P \,.\, K_D \to K_D \to k_P \to 0 ,$$

it is easy to see that P is a common zero of the global sections of K_D if and only if $H^1(D, m_PK_D) \to H^1(D,K_D)$ is not injective and thus by duality if and only if $0 \to H^0(D,\mathcal{O}_D) \to H^0(D,\mathcal{O}_D(P))$ is not surjective. Since D is 1-connected $h^0(D,\mathcal{O}_D) = 1$ (by 3.3), hence the assertion.

(6.2) Lemma Let D be a 1-connected divisor on a surface with $p_a(D) \geq 1$. If P is a non-singular point of D which is a common zero of the global sections of K_D then

(i) D is not 2-connected.

(ii) The unique component Γ of D containing P is isomorphic to $\mathbb{P}^1$ and every $s \in H^0(D,K_D)$ vanishes along Γ, that is $s_{|\Gamma} \equiv 0$.

Proof The result follows immediately from (6.1) and (3.5).

(6.3) Proposition If D is a 2-connected divisor on a surface either $D \cong \mathbb{P}^1$ or K_D is generated by its global sections.

Furthermore, if $p_a(D) = 1$ then $K_D \cong \mathcal{O}_D$.

Proof The first part of the assertion follows trivially from (5.3) and (6.2) and the second part from (3.3).

(6.4) Definition Given an effective divisor D on a surface, K_D is nef (numerically effective) if $\deg K_{D|\Gamma} \geq 0$ for all $\Gamma \subset D$.

(6.5) Proposition If D is a 1-connected divisor and K_D is not nef then either

(i) D is irreducible and isomorphic to $\mathbb{P}^1$

or

(ii) D is reducible and has a component $\Gamma \cong \mathbb{P}^1$ such that $\Gamma(D-\Gamma) = 1$.

In particular if D is 2-connected then either $D \cong \mathbb{P}^1$ or K_D is nef.

Proof If D is irreducible there is nothing to prove. If D is reducible the result follows immediately from 1-connectedness and the adjunction formula:

$$\deg K_{D|_C} = \deg K_C + C(D-C)\,, \quad \text{for a component } C \text{ of } D\,.$$

(6.6) Proposition Let D be an effective 1-connected divisor with $p_a(D) \geq 2$. Then K_D is ample if either

(i) D is irreducible

or

(ii) D is reducible and $C(D-C) \geq 3$ for every component $C \cong \mathbb{P}^1$ of D.

Proof If D is irreducible the result is trivial, since $\deg K_D > 0$. Otherwise K_D is ample if and only if $\deg_C K_D > 0$ for every component C of D, and the statement follows from $\deg K_{D|_C} = \deg K_C + (D-C)C$.

(6.7) Lemma Let D be a 1-connected divisor on a surface, with $p_a(D) \geq 2$ and K_D nef. Then

(i) $h^1(D, nK_D) = 0$, for $n \geq 2$.

(ii) If $D_1 \subset D$ is such that $0 < D_1 < D$ then $h^1(D_1, nK_D) = 0$ for $n \geq 1$.

Proof By duality (1.5),

$$h^1(D, nK_D) = h^0(D,(1-n)K_D) \text{ and}$$
$$h^1(D_1, nK_D) = h^0(D_1, K_{D_1} - nK_D)\,.$$

Since $p_a(D) \geq 2$, $\deg_D \mathcal{O}_D((1-n)K_D) < 0$ for $n \geq 2$. Since K_D is nef, $\deg_\Gamma \mathcal{O}_D((1-n)K_D) \leq 0$ for every $\Gamma < D$. Hence (i) follows by (3.2).

For (ii) we remark that $\deg_{D_1}(K_{D_1}-nK_D) = D_1(-(D-D_1)-(n-1)K_D)$. Since K_D is nef and D is 1-connected, this degree is negative. Also, for every $\Gamma \subset D$ we have

$$\Gamma(D_1-\Gamma) + \Gamma(D-D_1) > 0$$

and $\deg_\Gamma(nK_D) \geq 0$, so that $\Gamma(D_1-\Gamma) > \deg_{D_1-\Gamma}(K_{D_1}-nK_D)$.

The result now follows from (3.1).

(6.8) Lemma Let D be a 1-connected divisor such that K_D is nef and let $D_1 < D$ (with $D_1 > 0$) be 1-connected. Then

(i) If $\deg_{D_1} K_D > 0$,

$$h^1(\mathcal{O}_{D_1}(nK_D - (D-D_1)) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases}.$$

(ii) If $\deg_{D_1} K_D = 0$ (and thus $p_a(D_1) = 0$, $D_1(D-D_1) = 2$), then $h^1(\mathcal{O}_{D_1}(nK_D-(D-D_1)) = 1$, for $n \geq 1$.

Proof By duality (1.5) $h^1(\mathcal{O}_{D_1}(nK_D - (D-D_1)) = h^0(\mathcal{O}_{D_1}(-(n-1)K_D))$. By assumption D_1 is 1-connected, hence $h^0(\mathcal{O}_{D_1}) = 1$ and so (i) and (ii) are true for $n = 1$.

If $\deg_{D_1} K_D > 0$, then $\deg_{D_1} \mathcal{O}_{D_1}(-(n-1)K_D) < 0$ for $n \geq 2$. Since K_D is nef, $\deg_C(-(n-1)K_D) \leq 0$ for every $C \subset D_1$. As D_1 is 1-connected the result follows from (3.2).

If $\deg_{D_1} K_D = 0$ then $\deg_C K_D = 0$ for every component C of D_1 (because K_D is nef), and thus by (3.4) $K_{D|D_1} \cong \mathcal{O}_{D_1}$, hence (ii).

(6.9) Proposition Let D be a 1-connected divisor with K_D nef and let D_1 be a divisor with $0 < D_1 < D$ such that $D-D_1$ is 1-connected.

Consider the restriction maps

$$r_n : H^0(\mathcal{O}_D(nK_D)) \to H^0(\mathcal{O}_{D_1}(nK_D)).$$

Then either

(i) $\deg_{D-D_1} K_D > 0$ and r_n is surjective for all $n \geq 1$

or

(ii) $\deg_{D-D_1} K_D = 0$, r_1 is surjective and $\operatorname{Im} r_n$ has codimension 1 in $H^0(\mathcal{O}_{D_1}(nK_D))$ for $n \geq 2$.

Proof From the decomposition sequence (1.7) of $D = D_1 + (D-D_1)$ we get the long exact sequence

$$0 \to H^0(\mathcal{O}_{D-D_1}(nK_D-D_1)) \longrightarrow H^0(\mathcal{O}_D(nK_D)) \longrightarrow H^0(\mathcal{O}_{D_1}(nK_D)) \longrightarrow$$
$$\longrightarrow H^1(\mathcal{O}_{D-D_1}(nK_D-D_1)) \longrightarrow H^1(\mathcal{O}_D(nK_D)) \longrightarrow H^1(\mathcal{O}_{D_1}(nK_D)) \to 0.$$

The result follows by applying (6.7), (6.8).

Remark For more facts about K_D for reduced divisors we refer to [C]

Section 7. 2-connected divisors of genus 1.

(7.1) We will need the following analysis in Chapter II, but it is also interesting as a very simple example of how to determine $R(D, L)$.

The following (and more) is proved in [R-1].

(7.2) **Proposition** Let D be a 2-connected divisor with $p_a(D) = 1$ and L a sheaf on D such that $\deg_D L > 0$ and $\deg_C L \geq 0$ for every component C of D.

If $\deg_D L \geq 2$ then L is generated by its global sections. If $\deg_D L = 1$, $L = \mathcal{O}_D(P)$ with P a non-singular point of D and $H^0(L)$ is generated by one section vanishing only at P.

Proof If D is irreducible the result is well known. Suppose that D is reducible. Since $K_D \cong \mathcal{O}_D$ by (6.3), for any component C of D, $\deg_C K_D = 0$, and so necessarily $C(D-C) = 2$, $C \cong \mathbb{P}^1$ and $(D-C)$ is 1-connected by (2.2), with $p_a(D-C) = 0$.

Let C be a component of D. From the decomposition sequence (1.7) of $D = C + (D-C)$ we have the exact sequence

$$H^0(D,L) \xrightarrow{r} H^0(C,L) \longrightarrow H^1(D-C,\ L-C_{|D-C}).$$

By Serre duality $h^1(D-C,\ L-C_{|D-C}) = h^0(D-C,\ K_D-L_{|D-C})$, and so r is surjective (and no point of C is a base point of $|L|$) unless $\deg L_{|D-C} = 0$. In this case C is the only component of D such that $\deg_C L \neq 0$ and appears with multiplicity 1 in D.

Under these conditions, if $P \in C$ is a base point of $|L|$ it must be non-singular in D (because if C′ is another component of D such that $\deg_{C'} L = 0$, the above argument shows that C′ is not a fixed component of L).

Studying the long exact sequence obtained from

$0 \to m_P L \longrightarrow L \longrightarrow k_P \to 0$ and using duality, it is clear that P is a base point of $|L|$ if and only if $H^0(\mathcal{O}_D(-L+P)) \neq 0$ and hence $\deg L_{|C} = 1$.

(7.3) **Proposition** Let D be a 2-connected divisor with $p_a(D) = 1$ and L a sheaf on D such that $\deg_D L > 0$ and $\deg_C L \geq 0$ for every component C of D.

(i) If $\deg L = 1$ then

$R(D,L) \cong k[X,Y,Z]/I$, with $\deg X = 1$, $\deg Y = 2$, $\deg Z = 3$,

and $I = (f_6)$ is a principal ideal generated by a weighted homogeneous polynomial f_6 of deg 6 in which both Z^2 and Y^3 appear with nonzero coefficients.

(ii) If $\deg L = 2$ then

$R(D,L) \cong k[X_1,X_2,Y]/I$ where $\deg X_1 = \deg X_2 = 1$, $\deg Y = 2$ and I is a principal ideal generated by a weighted homogeneous polynomial f_4 of deg 4, in which Y^2 appears with nonzero coefficient.

Proof By Serre duality and (3.2), $h^1(nL) = 0$ for $n \geq 1$ and so by Riemann-Roch $h^0(nL) = nd$, where $d = \deg_D L$.

(i) Let us remark that this proof is the same as the classical argument for the Weierstrass normal form of an elliptic curve.

Suppose that $d = 1$. Then by (7.2), $H^0(2L)$ is generated by its global sections. By the free pencil trick the maps

$$\varphi_{2,n} : H^0(2L) \otimes H^0(nL) \to H^0((n+2)L)$$

are surjective for $n \geq 3$ and $\operatorname{Im} \varphi_{2,1}$ has codimension 1 in $H^0(3L)$.

By (7.2), $H^0(L)$ is generated by a single element x, which only vanishes at a non-singular point P of D, and thus $x^2 \in H^0(2L)$ is nonzero. Choose $y \in H^0(2L)$ such that (x^2,y) form a basis of $H^0(2L)$ as a k-vector space. Since $2L$ is generated by its global sections $y(P) \neq 0$. Similarly let $z \in H^0(3L)$ be such that (x^3,x^2y,z) form a basis of $H^0(3L)$. Again $z(P) \neq 0$.

Let $\varphi_{1,3} : H^0(L) \otimes H^0(3L) \to H^0(4L)$ .Since x vanishes only at P , Im $\varphi_{1,3}$ has codimension 1 in $H^0(4L)$. As $y^2(P) \neq 0$, $y^2 \notin \operatorname{Im} \varphi_{1,3}$ and hence $\operatorname{Im} \varphi_{1,3} + \operatorname{Im} \varphi_{2,2} = H^0(4L)$. Thus since $\varphi_{2,n}$ is surjective for $n \geq 4$, $R(D,L)$ is generated by x, y, z .

Let k[X,Y,Z] be the weighted polynomial ring with deg X = 1 , deg Y = 2 and deg Z = 3 and $\Delta : k[X,Y,Z] \to R(D,L)$ the homomorphism of graded rings sending $X \to x$, $Y \to y$, $Z \to z$. Then Δ is surjective by the above argument, and a simple dimension count shows dim $(\operatorname{Ker} \Delta)_n = 0$ for $n \leq 5$ and dim $(\operatorname{Ker} \Delta)_6 = 1$. If $f \in (\operatorname{Ker} \Delta)_6$ then Z^2 must appear in f with nonzero coefficient since $\operatorname{Im} \varphi_{2,4} = H^0(6L)$.

Then in R(D,L) we have a relation of the form $z^2 = \ldots$. Since y^3 and z^2 are the only elements of deg 6 which do not vanish at P , y^3 must also appear with nonzero coefficient.

Comparing $h^0(nL)$ with the degree n homogeneous piece of k[X,Y,Z]/(f) , by a simple dimension count, it is easy to see that $\ker \Delta = (f)$ (ideal generated by f), and hence

$$R(D, L) \cong k[X,Y,Z]/(f) \text{ with } \deg X = 1 , \deg Y = 2 , \deg Z = 3 .$$

(ii) Suppose that d = 2 . Then L is generated by its global sections and the free pencil trick shows that the maps

$$\varphi_{1,n} : H^0(L) \otimes H^0(nL) \to H^0((n+1)L)$$

are surjective for $n \geq 2$ and $\mathrm{Im}\,\varphi_{1,1}$ has codimension 1 in $H^0(2L)$. Choose x_1, x_2 such that (x_1,x_2) form a basis of the k-vector space $H^0(L)$ and $y \in H^0(2L)$ such that $(x_1^2, x_1x_2, x_2^2, y)$ form a basis of $H^0(2L)$. Then $R(D,L)$ is generated by x_1, x_2, y.

Consider $\Delta : k[X_1,X_2,Y] \to R(D,L)$ (with $\deg X_1 = \deg X_2 = 1$, $\deg Y = 2$). As in (i), $(\ker \Delta)_n = 0$ for $n \leq 3$ and $\dim(\ker \Delta)_4 = 1$. Since $\mathrm{Im}\,\varphi_{1,3} = H^0(4L)$ necessarily Y^2 appears with nonzero coefficient in $f \in (\ker \Delta)_4$. Again by a simple dimension count it is easy to see that $\ker \Delta = (f)$, and hence

$$R(D,L) = k[X_1,X_2,Y]/(f) \text{ with } \deg X_i = 1\text{, } \deg Y = 2.$$

(7.4) **Remark** By a suitable change of coordinates (completing the square and the cube), we can assume that the relations in (i) and (ii) are of the form

$$Z^2 = Y^3 + aYX^4 + bX^6$$

and $\quad Y^2 - a_0 X_1{}^4 - \dots - a_4 X_2{}^4$.

Section 8. Fibrations.

For all the assertions made in this section without proof refer to [B-P-V] Chapter III and [XG], §1.

(8.1) Let S be a smooth surface over $\mathbb{C}$. A *fibration* f of S is a proper surjective morphism $f : S \longrightarrow B$ with connected fibres, where B is a smooth projective curve.

(8.2) Almost all the fibres of a fibration are smooth and have the same genus; we say that f is a *genus g fibration* if the genus of a general fibre is g.

(8.3) A fibration is *relatively minimal* if no fibre F ($F = f^*(p)$, with $p \in B$) contains a (-1)-curve.

(8.4) It is well known that for $g \geq 1$ a fibration has a unique relatively minimal model, and we will only consider relatively minimal fibrations in what follows.

(8.5) If F is a fibre then the normal bundle $\mathcal{O}_F(F)$ is trivial.

(8.6) If F is a reducible fibre, $F = \Sigma\, n_i C_i$, with $C_i \subset S$ irreducible curves and $n_i > 0$, then

(i) $C_i F = 0$

(ii) if $D = \Sigma\, m_i C_i$, with $m_i \in \mathbb{Z}$, then $D^2 \leq 0$ and equality holds if and only if $D = aF$, with $a \in \mathbb{Q}$.

In other words the intersection form on the components of F is negative semidefinite with 1-dimensional kernel.

(8.7) In particular if F is a fibre, then either F is 1-connected or $F = nD$, where D is a 1-connected divisor with $D^2 = 0$. In this last case F is called a *multiple fibre* of *multiplicity* n.

(8.8) If $F = nD$ is a multiple fibre of multiplicity n then $\mathcal{O}_D(D)$ is a torsion sheaf of order n. Also if $p_a(F) = g$ then n divides $g - 1$. (See [XG] p.1

and 2).

(8.9) By base change results, $\chi(\mathcal{O}_F) = 1-g$ is constant. Furthermore, for every fibre F, $h^0(F, \mathcal{O}_F) = 1$ and $h^1(\mathcal{O}_F) = g$ and thus by duality, in a genus g fibration $h^0(F, K_F) = g$, for every fibre F.

Notice that for (8.8) and (8.9) the assumption of characteristic 0 is essential (see [XG] p.1 and 2).

(8.10) If f is a fibration of genus $g \geq 1$ and C is a curve appearing in a fibre F of f then $K_S C \geq 0$ (since we are only considering relatively minimal fibrations). If $g \geq 2$ then $K_S C = 0$ if and only if C is a (-2)-curve.

(8.11) If $f : S \to B$ is a fibration the *relative dualizing sheaf* ω_f of f is the invertible sheaf $\omega_{S/B} = K_S \otimes f^*(K_B{}^\vee)$ and the *relative canonical algebra* of f is

$$\mathcal{R}(f, \omega_f) = \bigoplus_{n\geq 0} f_* \omega_{S/B}^{\otimes n} .$$

(8.12) If F is a fibre of f, $\omega_{S/B|_F} = K_F$ by adjunction. Then by (8.9) and well-known results of base change (see [X], [B-P-V], [H] Ch.III-section 12) $f_*\omega_{S/B}^{\otimes n}$ on B is locally free and commutes with base change. Furthermore for every point $P \in B$ and for every $n \in \mathbb{N}$

$$f_*\omega_{S/B}^{\otimes n} \otimes k(P) \longrightarrow H^0(nK_F) \quad (\text{where} \quad F = f^*(P) \quad \text{and}$$

$k(P) = \mathcal{O}_B/m_P)$ is an isomorphism.

So, by Nakayama's lemma, results about the generation of $R(F, K_F)$ will imply the same results for $\mathcal{R}(f, \omega_f)$.

(8.13) Proposition Let F be a fibre of a fibration of genus $g \geq 2$. Then $2K_F$ is generated by its global sections.

Proof Since $\mathcal{O}_F(F) = \mathcal{O}_F$ it is enough to prove that for some $n \in \mathbb{N}$ the linear system $|2K_S + nF|$ has no base points in F.

As $p_a(F) \geq 2$ and $F^2 = 0$, $K_SF > 0$, and thus $(K_S + nF)^2 > 0$ for $n >> 0$.

By the Riemann-Roch theorem we have

$$h^0(K_S+nF) - h^1(-nF) + h^0(-nF) = \chi(0_S) + \tfrac{1}{2}(nKF)$$

and thus for $n >> 0$, $h^0(K_S + nF) > 0$.

Also, if $E \in |K_S + nF|$ is an effective divisor, $CE \geq 0$ for any curve C in S. In fact, since F is nef $CF \geq 0$ for any curve C in S. If $KC < 0$ and $C^2 < 0$, C is necessarily a (-1)-curve. Since we are considering relatively minimal fibrations C is not contained in any fibre F'. Hence $CF > 0$.

So there exists n_1 such that for $n \geq n_1$

(i) $h^0(S, K_S + nF) > 0$,

(ii) $(K_S + nF)^2 > 0$,

(iii) $(K_S + nF)$ is nef.

Let n_0 be such that $K_S + n_0F$ satisfies (i) (ii) (iii) and

(iv) $n_0 \equiv 0 \pmod 2$.

(v) Assume moreover that there exists $A \in |K_S + n_0F|$ having multiplicity bigger than 2 at each point of F.

Let P be any point of F, $\pi : \tilde{S} \to S$ the blow-up at P, and $E = \pi^{-1}(P)$

the exceptional divisor

By (i) $|K_S + n_0F| \neq \emptyset$ and using (4.6) any $A \in |K_S + n_0F|$ is 2-connected.

By (v) π^*A-2E is an effective divisor and by (4.5), π^*A-2E is 1-connected and hence by Francia's lemma (4.3) and (4.2)

$$h^1(\tilde{S}, -(\pi^*A-2E)) = h^1(\tilde{S}, -(\pi^*A-E)).$$

By duality $h^1(\tilde{S}, -(\pi^*A-2E)) = h^1(\tilde{S}, \pi^*(2K_S + n_0F)-E)$

and $h^1(\tilde{S}, -(\pi^*A-E)) = h^1(\tilde{S}, \pi^*(2K_S + n_0F))$

and so by Lemma, (4.4) P is not a base point of $|2K_S + n_0F|$.

Section 9. Some general results on $R(D, K_D)$

(9.1 The main aim of this section is to give bounds on the degrees of the generators of the ring $R(F, K_F)$ for F a fibre with $p_a(F) \geq 2$ (see Section 8). In some cases we do this in a slightly more general context by giving bounds for the generators of $R(D, K_D)$, for D a divisor on a surface satisfying certain conditions.

Properties of K_D and $R(D, K_D)$ are studied in great detail in [C], for D a reduced divisor (and more generally a Gorenstein curve). We will not, however, use the results of [C].

If D is an irreducible curve with $g = p_a(D) \geq 3$ it is well-known (see [S], [R-2], [C]) that $R(D, K_D)$ is generated by its elements of degree 1 if D is not hyperelliptic and by its elements of degrees 1 and 2 if D is hyperelliptic.

We describe in (9.4) the ring $R(D, K_D)$ for a 2-connected genus

2 divisor.

We will need to consider reducible divisors and we will use the following:

(9.2) **Fact** Let D be a reducible divisor, D = A + B a decomposition of D and L an invertible sheaf on D. The decomposition sequence (1.7) of D = A + B gives rise to exact sequences

$$0 \longrightarrow H^0\big(A, \mathcal{O}_A(L-B)\big) \longrightarrow H^0(D, L) \longrightarrow H^0(B, L)$$

and if M is another invertible sheaf on D we have the following commutative diagram with exact top and bottom rows:

$$\begin{array}{ccccccc}
0 \to & H^0(A,L-B)\otimes H^0(D,M) & \to & H^0(D,L)\otimes H^0(D,M) & \to & H^0(B,L)\otimes H^0(D,M) \\
& \downarrow \alpha_1 & & & & \downarrow \gamma_1 \\
& H^0(A,L-B)\otimes H^0(A,M) & & \downarrow \beta & & H^0(B,L)\otimes H^0(B,M) \\
& \downarrow \alpha_2 & & & & \downarrow \gamma_2 \\
0 \to & H^0(A,L+M-B) & \longrightarrow & H^0(D,L+M) & \longrightarrow & H^0(B,L+M) & .
\end{array}$$

From the diagram it is clear that β is surjective provided the restriction maps

$$\begin{cases} H^0(D, L) \to H^0(B, L) \\ H^0(D, M) \to H^0(B, M) \\ H^0(D, M) \to H^0(A, M) \end{cases}$$

and the multiplication maps γ_2 and α_2 are all surjective. So to show that β is surjective it is enough to show that these five maps are surjective. In general it is easier to do that.

Similarly if we want to show that $H^0(mL + nM)$ is generated by the sum of the images of

$$H^0\big((m-k)\,L + (n-p)M\big) \otimes H^0(kL + pM\big)$$

for k, p in the ranges $1 \le k < m$, $1 \le p < n$, and the appropriate restriction maps are surjective, it is enough to prove the same statement for $H^0(A, mL + nM\text{-}B)$ and $H^0(B, mL + nM)$.

(9.3) Proposition Let D be an effective divisor on a surface with $p_a(D) \ge 2$, such that $h^1(nK_D) = 0$ for $n \ge 2$. Then

(i) If $2K_D$ is generated by its global sections then $R(D, K_D)$ is generated as a k-algebra by its elements of degree ≤ 5.

(ii) If D is 2-connected then $R(D, K_D)$ is generated by its elements of degree ≤ 3.

Proof If $2K_D$ is generated by its global sections and $h^1(nK_D) = 0$, $n \ge 2$, by Castelnuovo's lemma (3.6) the maps

$$H^0(D, 2K_D) \otimes H^0(D, nK_D) \longrightarrow H^0(D, (n+2)K_D)$$

are surjective for $n \ge 4$, hence (i).

If D is 2-connected, by (6.3) K_D is generated by its global sections and thus, since $h^1(nK_D) = 0$ for $n \ge 2$, the maps $H^0(D, K_D) \otimes H^0(D, nK_D) \to H^0(D, (n+1)K_D)$ are surjective for $n \ge 3$, hence (ii).

(9.4) Proposition Let D be a 2-connected divisor with $p_a(D) = 2$. Then $R(D, K_D) = k[X_0, X_1, Z] / (f_6)$, with $\deg X_1 = \deg X_2 = 1$, $\deg Z = 3$ and f_6 a weighted homogeneous polynomial of degree 6, where Z^2 appears with nonzero coefficient.

Furthermore f_6 can be chosen to be $Z^2 - p_6(X_0, X_1)$.

Proof This proof is similar to the proof of (7.3) and so we skip the details.

By (6.3) K_D is generated by its global sections and by (6.7), $h^1(nK_D) = 0$, for $n \geq 2$. Since $h^0(K_D) = 2$, using Riemann-Roch and the free pencil trick we see that the maps $\varphi_m : H^0(K_D) \otimes H^0(mK_D) \longrightarrow H^0((m+1)K_D)$, are surjective for $m \neq 2$ and $\operatorname{Im} \varphi_2$ has codimension 1. Then using an argument similar to the one used in (7.3) (ii), we have $R(D, K_D) = k[X_0, X_1, Z]/(f_6)$ with $\deg X_0 = \deg X_1 = 1$, $\deg Z = 3$ and f_6 a polynomial where Z^2 appears with nonzero coefficient.

The fact that f_6 can be chosen to be $Z^2 - f'_6(x_0, x_1)$ is just the standard trick of "completing the square".

(9.5) Lemma Let D be an effective m-connected divisor (with $m \geq 1$) such that D is nef. If $D_1 \subset D$ is minimal subject to $D_1(D-D_1) = m$, then $h^0(D_1, 2K_{D_1}-2K_D) = 0$.

Proof By (2.2) and (2.3) both D_1 and $D-D_1$ are 1-connected, and either $D_1 \subset D-D$, or D_1 and $D-D_1$ have no common components. Furthermore for every $\Gamma \subset D$, $\deg_\Gamma (2K_{D_1}-2K_D) = -2\Gamma(D-D_1)$ and thus $D_1(2K_{D_1}-2K_D) =$ $= -2m < 0$. Let $D-D_1 = D_2$.

If $h^0(D, 2K_{D_1}-2K_D) \neq 0$, by (3.1) there exists a decomposition of D_1, $D_1 = A+B$ such that $AB \leq -2AD_2$. Since D_1 is 1-connected necessarily $AD_2 < 0$ and thus $D_1 \subset D_2$. But then we have $AB + AD_2 + A^2 \leq - A(D_2-A)$ which is impossible because the left-hand side is AD which is greater than 0 by assumption and $A(D_2-A) \geq 1$ by 1-connectedness of D_2. So $h^0(D_1, 2K_{D_1} - 2K_D) = 0$.

(9.6) Proposition If F is a 1-connected fibre with $p_a(F) \geq 2$, then $R(F, K_F)$ is generated by its elements of degree ≤ 4.

Proof

Step 1 Since we are considering only relative minimal fibrations, K_F is nef and hence by (6.7) $h^1(nK_F) = 0$ for $n \geq 2$. By (8.13), $2K_F$ is generated by its global sections and thus by (9.3) we only have to show that $H^0(2K_F) \otimes H^0(3K_F) + H^0(K_F) \otimes H^0(4K_F) \to H^0(5K_F)$ is surjective if F is 1-connected and not 2-connected (i.e. F decomposes as $F = D_1 + D_2$ with $D_1 D_2 = 1$).

Let $F = D_1 + D_2$ be a decomposition of F such that D_1 is minimal with respect to $D_1(F-D_1) = 1$. By (2.2), D_1 is 2-connected and D_2 is 1-connected and by (2.3), either D_1 and D_2 have no common components or $D_1 \subset D_2$.

Since K_F is nef and for $i = 1,2$, we have $\deg_{D_i} K_F = \deg K_{D_i} + 1$, and so $p_a(D_i) > 0$ and $\deg_{D_i} K_F > 0$. Thus by (6.9), the restriction maps

$$\varphi_{n,i} : H^0(F, nK_F) \to H^0(D_i, nK_F)$$

are surjective for $i = 1,2$.

Note that $\operatorname{Ker} \varphi_{1,2} = H^0(D_1, K_{D_1})$. By (9.2), using the fact that the maps $\varphi_{n,i}$ are surjective, to prove the statement it is enough to show that the maps

$$\textbf{(A)} \quad H^0(D_2, 2K_F) \otimes H^0(D_2, 3K_F) \to H^0(D_2, 5K_F)$$

and

$$\textbf{(B)} \quad H^0(D_1, K_{D_1}) \otimes H^0(D_1, 4K_D) \to H^0(D_1, 4K_D + K_{D_1})$$

are surjective.

Step 2. Proof of (A) and (B)

(A) follows trivially from Castelnuovo's lemma since by (8.13) $2K_F$ is generated by its global sections and by (6.7) $h^1(D_2, K_F) = 0$.

(B) will also follow from Castelnuovo's lemma if we show that $h^1(D_1, 4K_F-K_{D_1}) = 0$, because, by (6.3), K_{D_1} is generated by its global sections.

By duality $h^1(D_1, 4K_F-K_{D_1}) = h^0(D_1, 2K_{D_1}-4K_F)$. Since $2K_F$ is generated by its global sections and by (9.5) the image of the map $H^0(D_1, 2K_{D_1}-4K_F) \otimes H^0(D_1, 2K_F) \longrightarrow H^0(D_1, 2K_{D_1}-2K_F)$ is 0, hence $h^1(D_1, 4K_F-K_{D_1}) = 0$.

(9.7) Remark Note that (9.6) is also true for an effective 1-connected divisor D with D reduced or D nef, $2K_D$ generated by its global sections and $h^1(nK_D) = 0$, for $n \geq 2$.

(9.8) Proposition Let D be an m-connected $(m \geq 2)$ effective divisor with $g = p_a(D) \geq 3$. If D admits a decomposition $D = D_1+D_2$ satisfying either:

(i) D_1 is 2-connected, $p_a(D_1) \geq 1$, D_2 is 1-connected, $\deg K_{D|D_1} > 0$, $\deg K_{D|D_2} > 0$ and $h^1(D_1, 2K_D-K_{D_1}) = 0$,

or

(ii) $D_1 \cong \mathbb{P}^1$, D_2 is 1-connected and $D_1D_2 \geq 4$,

then $R(D, K_D)$ is generated by its elements of degree lesser or equal to 2.

Proof If D has a decomposition as in (i) or (ii) of (9.8), using the same reasoning as in step 1 of (9.6), we see that to prove the statement it will be enough to show that

(A) $H^0(D_2, K_D) \otimes H^0(D_2, 2K_D) \to H^0(D_2, 3K_D)$ is surjective

and

(B) $H^0(D_1, K_{D_1}) \otimes H^0(D_1, 2K_D) + H^0(D_1, K_D) \otimes H^0(D_1, K_D+K_{D_1}) \to$
$\to H^0(D_1, 2K_D+K_{D_1})$ is surjective.

Now using the facts that K_D is generated by its global sections and $h^1(D_2, K_D) = 0$, (A) comes from Castelnuovo's lemma (as in step 2 of (9.6)).

For (B) if the decomposition of D is as in (i) of (9.8) again by Castelnuovo's lemma

$$H^0(D_1, K_{D_1}) \otimes H^0(D_1, 2K_D) \longrightarrow H^0(D_1, 2K_D + K_{D_1})$$

is surjective.

If the decomposition of D is as in (ii) of (9.8) then $D_1 \cong \mathbb{P}^1$, $\mathcal{O}_{D_1}(K_D + K_{D_1}) = \mathcal{O}_{D_1}(m-4)$, $\mathcal{O}_{D_1}(K_D) = \mathcal{O}_{D_1}(m-2)$ and so

$$H^0(D_1, K_D + K_{D_1}) \otimes H^0(D_1, K_D) \longrightarrow H^0(D_1, 2K_D + K_{D_1})$$

is surjective.

So (B) is proved for both cases.

(9.9) Corollary If F is an m-connected fibre ($m \geq 4$) with $g=p_a(F) \geq 3$ then $R(F, K_F)$ is generated by its elements of degree lesser or equal to 2.

Proof The statement is true for irreducible fibres ,so we only have to consider reducible fibres. It is easy to check that ,since $m \geq 4$ any decomposition $F= D_1+D_2$ with D_1 minimal with respect to $D_1(F-D_1)=m$ is as in (i) or (ii) of (9.8),hence the statement.

(9.10) Proposition If F is a 2-connected fibre with $p_a(F) = 3$ then $R(F, K_F)$ is generated by its elements of degree lesser or equal to 2.

Proof By (9.9) we only have to prove the statement if F is not 4-connected.In this case, using the numerical properties of F (8.6) , (I.2.2) and the fact that for a 2-connected genus 1 divisor $K_D \cong \mathcal{O}_D$, it is not very difficult to see that F has a decomposition as in (9.8) unless F contains a unique curve C with $KC \neq 0$.Then $KC=4$ and all the other components of F are (-2)-curves.

In this case the statement can be proved as in the irreducible case.

CHAPTER II

Results on fibres

Section 1. Lifting R/x_0R to R .

(1.0) In this chapter and the next we study the ring $R(F,K_F)$ for F a fibre with $p_a(F) = 2$ or 3 .

The principal method for dealing with reducible fibres (not 2-connected)is to start by describing a homomorphic image of $R(F, K_F)$ obtained by restricting K_F to some divisor $D \subset F$, and then recovering $R(F, K_F)$. For this we will repeatedly need to use the general theory worked out in this section.

(1.1) Let R be a graded k-algebra and $x_0 \in R$ a homogeneous element of degree 1 . Suppose that there exists an isomorphism of graded rings

$$S = R/x_0R \cong k[X_1, \dots, X_n]/I,$$

where $k[X_1, \dots, X_n]$ is a weighted polynomial ring and I a homogeneous ideal. Let $k[X_0, \dots, X_n] = k[X_1, \dots, X_n][X_0]$ with weight $X_0 = 1$.

Then the isomorphism between R/x_0R and $k[X_1, \dots, X_n] / I$ can be covered by a graded epimorphism $\varphi : k[X_0, \dots, X_n] \to R$ taking X_0 to x_0 . This gives rise to the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (X_0)\cap\tilde{I} & \longrightarrow & \mathrm{Ker}\,\varphi = \tilde{I} & \xrightarrow{\alpha|_{\tilde{I}}} & I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (X_0) & \longrightarrow & k[X_0,\dots,X_n] & \xrightarrow{\alpha} & k[X_1,\dots,X_n] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow\varphi & & \downarrow & & \\
0 & \longrightarrow & x_0R & \longrightarrow & R & \longrightarrow & S=k[X_1,\dots,X_n]/I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where α is the natural quotient map (sending X_0 to 0) from $k[X_0,\dots,X_n]$ to $k[X_1,\dots,X_n]$ and $\tilde{I}$ is a homogeneous ideal. The surjectivity of $\alpha|_{\tilde{I}}$ comes from the snake lemma.

(1.2) Notation Elements of $k[X_0,\dots,X_n]$ will be denoted by capital letters and their images in R by the corresponding small letters. For $F \in k[X_1,\dots,X_n]$ we will also denote by F its image $i(F)$, where $i : k[X_1,\dots,X_n] \to k[X_0,\dots,X_n]$ is the inclusion map. For a graded ring A we denote by A_q its homogeneous piece of degree q.

(1.3) Let $F_1,\dots,F_m$ be a set of homogeneous generators of I as an ideal, where $F_i \in k[X_1,\dots,X_n]_{q_i}$. Since $\alpha : \tilde{I} \to I$ is surjective, for each $i = 1,\dots,m$ there exists $G_i \in k[X_0,X_1,\dots,X_n]_{q_i-1}$ such that $\tilde{F}_i = F_i - X_0G_i \in \tilde{I}_{q_i}$.

(1.4) Let $J = \{y \in R : x_0 y = 0\}$. J is the kernel of the linear map of degree 1 of graded algebras $R \to R$ defined by $y \to x_0 y$ and so is a homogeneous ideal of R. Let $p_1, \ldots, p_\ell$ be homogeneous elements of J such that $J = (p_1, \ldots, p_\ell)$ (i.e. $J = \sum_{i=1}^{\ell} k[X_0, \ldots, X_n] p_i$). For $j = 1, \ldots, \ell$ let $P_1, \ldots, P_\ell$ be any set of homogeneous elements of $k[X_0, \ldots, X_n]$ such that $\varphi(P_j) = p_j$. Then we have

(1.5) **Lemma** $\tilde{I}$ is the ideal generated by $\{X_0 P_1, \ldots, X_0 P_\ell, \tilde{F}_1, \ldots, \tilde{F}_m\}$.

Proof Let $\tilde{I}_q$ be the homogeneous part of degree q so that $\tilde{I} = \sum_{q=0}^{\infty} \tilde{I}_q$. We prove that any $F \in \tilde{I}_q$ can be written in the form

$$F = \sum_{i=1}^{m} H_i \tilde{F}_i + X_0 \sum_{j=1}^{\ell} B_j P_j + X_0 F' \text{ with } F' \in \tilde{I}_{q-1}.$$

The result will then follow by induction.

For $F \in \tilde{I}_q$, $\alpha(F) \in I_q$, and so $\alpha(F) = \sum_{i=1}^{m} H_i F_i$, with $H_i \in k[X_1, \ldots, X_n]$ and $H_i F_i \in I_q$. Then (considering H_i as an element of $k[X_0, \ldots, X_n]$),

$$F - \sum_{i=1}^{m} H_i \tilde{F}_i \in (\operatorname{Ker} \alpha_{\tilde{I}})_q = \left((X_0) \cap \tilde{I}\right)_q.$$

So $F - \sum_{i=1}^{m} H_i \tilde{F}_i = X_0 \Delta$ with $\Delta \in k[X_0, \ldots, X_n]_{q-1}$ and $\varphi(X_0 \Delta) = 0$.

Then $\varphi(\Delta) = \delta \in J_{q-1}$, and hence $\varphi(\Delta) = \sum_{j=1}^{\ell} B_j p_j$ with $B_j \in k[X_0, \ldots, X_n]$ and $B_j p_j \in R_{q-1}$.

$$\text{So}\quad \Delta - \sum_{j=1}^{\ell} B_jP_j \in k[X_0,\dots,X_n]_{q-1} \cap \tilde{I} = \tilde{I}_{q-1}\,. \qquad \text{Hence}$$

$$F = \sum_{i=1}^{m} H_i\tilde{F}_i + X_0 \sum_{j=1}^{\ell} B_jP_j + X_0F' \quad \text{with } F' \in \tilde{I}_{q-1}\,.$$

(1.6) Remark In view of this lemma, as long as we know how I is generated, we can determine $\tilde{I}$ by finding polynomials G_i such that $F_i - X_0G_i \in \tilde{I}_{q_i}$, for $i = \{1,\dots,m\}$ and polynomials $P_1,.. P_\ell$ such that $(\varphi(P_1) = p_1, \dots, \varphi(P_\ell) = p_\ell) = J$.

In the cases where we will be using this we will have additional specific information, namely that

(A) J is also the kernel of a surjective morphism of graded rings $\psi : R \to k[X, Y, Z]/(F_6)$ where $k[X, Y, Z]$ is the graded polynomial ring with $\deg X = 1$, $\deg Y = 2$, $\deg Z = 3$ and $F_6 \in k[X, Y, Z]_6$ is of the form $Z^2 - \lambda Y^3 - XP_5$, with $\lambda \neq 0$ and $P_5 \in k[X, Y, Z]_5$ (this ring is as in (I.7.3,(i)).

(B) J is generated by its homogeneous elements of degree < 6.

(1.7) Under the conditions of (1.6) there exist lifts of X, Y, Z to R, i.e. $x \in R_1$, $y \in R_2 \setminus R_1R_1$, $z \in R_3 \setminus R_1R_2$ such that $\psi(x) = X + (F_6)$, $\psi(y) = Y + (F_6)$, $\psi(z) = Z + (F_6)$. Let $A = k[x, y, z] \subset R$. $\psi_{|A}$ is still an epimorphism and so for each $f \in R_q$ there exists $a \in A_q$ such that $f - a \in (\operatorname{Ker}\psi)_q = J_q$.

In fact by changing generators in $k[X, Y, Z]/(F_6)$ we can assume that $\{x, y, z\} \subset \{x_0, \dots, x_n\}$, i.e. there exist $\{i, j, k\} \subset \{0,\dots,n\}$, such that $x = x_i$, $y = x_j$ and $z = x_k$. For each x_ℓ, with $\ell \in \{0,\dots, n\} \setminus \{i,j,k\}$ if $\deg x_\ell = d_\ell$ then there exists $a_\ell \in A_{d_\ell}$ such that $x_\ell - a_\ell \in J_{d_\ell}$.

(1.8) **Lemma** Under the conditions of (1.6) and (1.7)

(i) $J = \sum_{\ell} k[X_0,\dots,X_n]\,(x_\ell - a_\ell)$, for each $\ell \in \{0,\dots,n\} \setminus \{i,j,k\}$

and

(ii) $x_0R = x_0(k[x_i,x_j] + x_k k[x_i,x_j])$.

Here i, j, k is the choice made in (1.7). Furthermore

(iii) $x_0(k[x_i,x_j] + x_k k[x_i,x_j])$ is isomorphic as a k-vector space to $k[X,Y] \oplus Zk[X,Y] \subset k[X,Y,Z]$, where $k[X,Y,Z]$ is as before (and so $X_0(k[X_i,X_j] + X_k k[X_i,X_j]) \cap \tilde{I} = \{0\})$.

Proof (i) From the definition of the ring A and property (A) we see that $A_q \cap J_q = \{0\}$, for $q < 6$. By assumption (B), J is generated by its elements of degree < 6 , and so to see that $J = \sum_{\ell} k[X_0,\dots,X_n]\,(x_\ell - a_\ell)$ it is enough to show that $\sum_{q=0}^{5} J_q \subset \sum_{\ell} k[X_0,\dots,X_n]\,(x_\ell - a_\ell)$.

Let $f \in J_q$, $q < 6$ and let $f = \sum_{\ell} \gamma_\ell x_\ell + a$,with $a \in A$. For each ℓ let b_ℓ be a homogeneous element of A such that $\gamma_\ell - b_\ell \in J$. Then we have

$$f - \sum_{\ell} \gamma_\ell(x_\ell - a_\ell) - \sum_{\ell} (\gamma_\ell - b_\ell)a_\ell \in A_q \cap J_q = \{0\} .$$

Thus $a = -\sum_{\ell} b_\ell a_\ell$ and $f = \sum_{\ell} \gamma_\ell\,(x_\ell - a_\ell) + \sum_{\ell} (\gamma_\ell - b_\ell)a_\ell$. Since $\gamma_\ell - b_\ell \in J$ and $\deg \gamma_\ell - b_\ell < q$, an inductive reasoning shows that $J_q \subset \sum_{\ell} (x_\ell - a_\ell)R$ for $q < 6$, and so we get (i). Now (ii) and (iii) are clear from property (A).

(1.9) Remark With this lemma and Lemma (1.5), if the conditions stated in (1.6) hold we can use the following procedure to describe $\tilde{I}$.

Step 1. Find $X_i, X_j, X_k \in k[X_0,..., X_n]$ such that $\psi(x_i)$, $\psi(x_j)$, $\psi(x_k)$ generate $k[X,Y,Z]/(F_6)$ as a k-algebra.

Step 2. For $\ell \in \{0,.. n\}\backslash\{i, j, k\}$ determine $A_\ell \in k[X_i,X_j,X_k] \subset k[X_0,...,X_n]$ such that $\psi(x_\ell - a_\ell) = 0$. Then by (1.8), $J = \sum_\ell k[X_0,..., X_n](x_\ell - a_\ell)$.

Step 3. Choose polynomials G_i such that $F_i - X_0 G_i \in \tilde{I}$. Since $\varphi(F_i) \in (x_0)$ and $(x_0) = x_0(k[x_i,x_j] + x_k k[x_i,x_j])$ the G_i can be chosen to belong to $k[X_i, X_j] + X_k k[X_i, X_j]$.

Explanation of Step 3. To determine the precise form of the G_i we will in most cases have to use the syzygies between the F_i in $k[X_1,..., X_n]$. In fact, since (x_0) is isomorphic as a k-vector space to $k[X,Y] + Zk[X,Y]$, it follows that $X_0(k[X_i,X_j] + X_k k[X_i,X_j]) \cap \tilde{I} = \{0\}$. If $\Sigma \Gamma_i F_i = 0$ is a syzygy in $k[X_1,.. X_n]$, then $\Sigma \Gamma_i \tilde{F}_i = \Sigma \Gamma_i F_i - X_0 \Sigma \Gamma_i G_i = -X_0 \Sigma \Gamma_i G_i \in (X_0) \cap \tilde{I}$. Since for any X_ℓ (with $\ell \notin \{i,j,k\}$) there exist $A_\ell \in k[X_i, X_j, X_k]$ such that $X_0(X_\ell - A_\ell) \in \tilde{I}$, it is possible to subtract suitable expressions from $X_0(\Sigma \Gamma_i G_i)$ in such a way that what is left $X_0(\Sigma \Gamma'_i G_i)$ still belongs to $\tilde{I}$ and $\Sigma \Gamma'_i G_i \in k[X_i, X_j, X_k]$. From this we can relate the coefficients of the different G_i and sometimes deduce relations between the A_ℓ.

Section 2. Elliptic tails in fibrations.

(2.1) **Terminology** In all this section fibre refers to a divisor on a surface appearing as a fibre of a relatively minimal fibration (I.8). Since $K_F = (K_S+F)_{|F} = K_{S_{|F}}$ we write K for it.

(2.2) The following definitions and lemmas will be needed to study fibres which are not 2-connected. The proofs only use the numerical properties of fibres namely (I.8.6) and (I.8.10).

(2.3) **Definition** Let F be a fibre. We say that $A \subset F$ is an *elliptic cycle* if $A^2 = -1$, $KA = 1$ .We also say that $E \subset F$ is an *elliptic tail* if E is a minimal elliptic cycle (i.e. if E does not contain another elliptic cycle).

(2.4) **Lemma** Let $A \subset F$ be an elliptic cycle. Then

(i) A is 1-connected, and

(ii) A is 2-connected if and only if A is an elliptic tail.

Proof If F contains an elliptic cycle A then $\deg K_F > 0$ and thus $p_a(F) \geq 2$

Let $A = A_1 + A_2$. From

$$A_1^2 + 2A_1A_2 + A_2^2 = A^2 = -1,$$

we have $A_1A_2 \geq 1$ and so A is 1-connected, proving (i). Also, if $A_1^2 < -1$, then $A_1A_2 > 1$ and thus an elliptic tail is 2-connected.

Now let $A_1 \subset A$ be an elliptic cycle. Since $(A + A_1)^2 \leq 0$, $A_1A \leq 1$ and thus $A_1(A - A_1) \leq 2$. So if A is 2-connected and $A-A_1 \neq 0$ the only possibility is $A_1(A-A_1) = 2$. Since $KA = KA_1 = 1$, $K(A-A_1) = 0$. By (I.2.2), $A-A_1$ is 1-connected and thus $(A-A_1)^2 = -2$. This gives $A^2 = -1 = A_1^2 + 2A_1(A-A_1) + (A-A_1)^2 > 0$, a contradiction. So a 2-connected elliptic cycle is an elliptic tail.

(2.5) Lemma Let A be an elliptic cycle and $E \subset A$ an elliptic tail. Then

(a) E is the unique elliptic tail contained in A. Moreover

(b) if $(A-E) \neq 0$ then either

(i) $AE = 1$ and $F = \lambda(A+E)$ with $\lambda = g-1$, where $g = p_a(F)$

or

(ii) $E(A-E) = 1$ and $(A-E)^2 = -2$.

Proof We start with the second part.

(b) $(A+E)^2 \leq 0$ and $(A+E)^2 = 0$ if and only if there exists $\lambda \in \mathbb{Q}$ such that $F = \lambda(A+E)$. In this case $\deg K_F = 2g-2$ and so $\lambda = g-1$. From $(A+E)^2 = 0$ we have $-1 + 2AE + (-1) = 0$ i.e. $AE = 1$. If $(A+E)^2 < 0$ then $AE \leq 0$. Since A is 1-connected, $EA = E^2 + E(A-E) \geq 0$ and so $AE = 0$ and $(A-E)^2 = -2$.

(a) Let E_1, E_2 be two elliptic tails contained in A . Since both E_1 , E_2 contain the unique component C_0 of A such that $KC_0 = 1$, we can write

$$E_1 = C + B_1$$
$$E_2 = C + B_2 .$$

where $C \neq 0$ and B_1, B_2 have no common components.

Let $A = C + B_1 + B_2 + D$; then D is made up of (-2)-curves, and by 2-connectedness of E_i , $\theta E_i \geq 0$ for every component θ of D .

Suppose that $B_1 \neq 0$ (and hence $B_2 \neq 0$) . By 2-connectedness of E_1

and E_2, we have $B_1E_2 \geq 2$, $B_2E_1 \geq 2$ and so

$$E_2 A = E_2{}^2 + E_2(B_1 + D) \geq 1 .$$

By (b), $F = \lambda(A+E_1) = \lambda(A+E_2)$, and hence

$$\lambda(A+B_1+C) = \lambda(A+B_2+C)$$

giving $B_1 = B_2$. This contradicts $B_1 \neq 0$, and B_1, B_2 have no common components. So $B_1 = 0 = B_2$ and thus $E_1 = E_2$, proving uniqueness.

(2.6) Description of elliptic tails. An irreducible elliptic tail E is either a non-singular elliptic curve or a rational curve with a cusp or a node.

If E is a reducible elliptic tail then all of its components are isomorphic to $\mathbb{P}^1$. Also E has one component C_0 appearing with multiplicity 1 such that $KC_0 = 1$ and all the other components are (-2)-curves.

By 2-connectedness of E, it follows that $C_0E \geq -1$ and $\theta E \geq 0$ for every (-2)-curve θ in E. Since $E^2 = -1$, necessarily $\theta E = 0$ and $C_0E = -1$. Thus $(E-C_0)^2 = -2$, and since $\theta C_0 \geq 0$, $\theta(E-C_0) \leq 0$ for every θ in $E-C_0$. So $E-C_0$ is a (-2)-cycle (see I.1). For a list of elliptic tails refer to [R-1].

(2.7) Lemma Let F be a 1-connected fibre with $g = p_a(F) \geq 3$, and let A_1, A_2 be two elliptic cycles contained in F. Then $A_1A_2 \leq 0$. Moreover if $A_2 < A_1$ and $A_2 - A_1 \neq 0$ then:

(i) $A_1 - A_2$ is a (-2)-cycle of type $\mathbf{A_n}$ and $\theta A_1 \leq 0$ for every θ in $A_1 - A_2$.

(ii) If A_2 is an elliptic tail then either A_2 and $A_1 - A_2$ have no common components or $A_1 - A_2 = \theta_0 + B$ where θ_0 appears with multiplicity 1 in A_1 and $B \subset A_2$, $\theta_0 B = \theta_0 A_2 = 1$.

Proof Since F is 1-connected and $p_a(F) \geq 3$, for any two elliptic cycles A, D in F, $AD \leq 0$. If $AD > 0$ then $(A+D)^2 = 0$, so $F = (g-1)(A+D)$. If $g \geq 3$ this contradicts 1-connectedness, and proves the first part.

If $D < A$ then by 1-connectedness of A, $DA = 0$ and $(A-D)^2 = -2$.

Consider now two elliptic cycles A_2, A_1 such that $A_2 \subset A_1$ and $A_1 - A_2 \neq 0$. Let $A' \subset F$ be a maximal elliptic cycle containing A_1 and let θ be a (-2)-curve appearing in A'. Then $\theta A' \leq 1$ (otherwise $(\theta + A')^2 > 0$). If $\theta A' = 1$ then $\theta(F - A') = -1$, hence $\theta \subset F - A'$, contradicting the maximality of A'. Thus for every (-2)-curve in A', $\theta A' \leq 0$.

Let E be the elliptic tail contained in A', ($E \subset A_2 \subset A_1$ by (2.5. (a)). By 2-connectedness of E, $\theta E \geq 0$, for every (-2)-curve θ in A', and so there exists a unique (-2)-curve θ_0 in A' such that $\theta_0 E = 1$, and for every other (-2)-curve θ in A', $\theta E = 0$.

But then, since $\theta_0 A' \leq 0$, $\theta_0(A' - E) < 0$, and thus for every θ in $(A'-E)$, we have $\theta(A'-E) \leq 0$. So $A' - E$ is a (-2)-cycle (1.8).

Since θ_0 appears with multiplicity 1 in $A'-E$ and $\theta_0(A'-E) < 0$, it follows that $A'-E$ must be of type $\mathbf{A_n}$. Since $A_1 - A_2 \subset A'-E$ and $(A_1-A_2)^2 = -2$, A_1-A_2 must be also of type $\mathbf{A_n}$.

Let $\theta \in A_1-A_2$. If $\theta A_1 > 0$ then necessarily (since A_1-A_2 is a (-2)-cycle) $\theta A_2 = 1$ and $\theta(A_1-A_2) = 0$. But then $D_1 = A_2 + \theta \subset A_1$ would be an elliptic cycle such that $D_1 A_1 \geq 1$, contradicting the remarks in the beginning of the proof. So (i) is proved.

(ii) Let A_2 be an elliptic tail contained in A_1. By the proof of (i), A_1-A_2 is a (-2)-cycle of type $\mathbf{A_n}$, and there exists a (-2)-curve θ_0 appearing in A_1 with multiplicity 1 such that $\theta_0 A_2 = 1$.

We can then write:

$$A_1 - A_2 = \theta_0 + B_1 + B_2$$

with $B_2 \subset A_2$ and B_1, B_2 without common components. Since $B_1 A_2 = 0$, B_1 is disjoint from A_2 and thus from B_2.

Since $\theta_0 A_1 \leq 0$, $\theta_0(B_1 + B_2) \leq 1$ and hence by 1-connectedness of $A_1 - A_2$ either $B_1 = 0$ or $B_2 = 0$, giving (ii).

(2.8) Lemma Let F be a 1-connected fibre and E a reducible elliptic tail appearing with multiplicity exactly m in F. If C_0 is the component of E such that $KC_0 = 1$ then $C_0 \not\subset F - mE$.

Proof Write $E = D + B$ and

$$F\text{-}mE = D + C$$

where B and C have no common components. Then $B \neq 0$ by assumption. Suppose $D \neq 0$ (otherwise there is nothing to prove). If $C_0 \subset D$ then $DE = -1$ and $BE = 0$. By 2-connectedness of E $BD \geq 2$ and so from

$$0 = BF = B\,(mE + D + C) = BD + BC$$

we have $BC < 0$, which is a contradiction, since by assumption B and C have no common components.

(2.9) Lemma Let F be a 1-connected fibre with $p_a(F) \geq 3$, and let A_1, A_2 be two elliptic cycles contained in F. Then either the two elliptic cycles are disjoint or one is contained in the other.

Proof By (2.7), $A_1 A_2 \leq 0$ and so if A_1, A_2 have no common components

A_1 and A_2 are disjoint.

Suppose now that A_1 and A_2 have common components.

Write $$A_1 = A + B$$ $$A_2 = A + C$$

where $A \neq 0$ and B,C have no common components, and let $F = A + B + C + D$

From $A_i(F-A_i) = 1$ for $i = 1,2$ we have

$$AC + AD + BC + BD = 1 \quad \text{and}$$

$$AB + AD + BC + CD = 1$$

and so $A(B + D + C) + D(A + B + C) + 2BC = 2$. Since $BC \geq 0$ and F is 1-connected, necessarily $A(B + D + C) = 1 = D(A + B + C)$ and $BC = 0$. Then $A^2 = -1$ (from $AF = 0$) and so $B^2 + 2AB = 0$ and $C^2 + 2AC = 0$.

Since $p_a(F) \geq 3$ and F is 1-connected we have $(2A + B + C)^2 < 0$ ($F = \lambda(2A+B+C)$ is only possible if either F is a multiple fibre or $\lambda = 1$ giving $p_a(F) = 2$).

But $(2A + B + C)^2 = 4A^2 + 2AB + 2AC + (B^2 + 2AB) + (C^2 + 2AC) =$ $= -4 + 2AB + 2AC$. Since both A_1 and A_2 are 1-connected, $-4 + 2AB + 2AC < 0$ is only possible if $B = 0$ or $C = 0$.

So either $A_1 \subset A_2$ or $A_2 \subset A_1$.

(2.10) Corollary If E_1, E_2 are two elliptic tails in a 1-connected fibre F with $p_a(F) \geq 3$, then either $E_1 = E_2$ or E_1 and E_2 are disjoint.

Section 3. Study of $R(F, K_F)$ for a genus 2 fibre.

In this section we are going to prove the following theorem which gives a complete description of the canonical ring of a genus 2 fibre. This result was pointed out to me by Miles Reid.

(3.1) Theorem Let F be a genus 2 fibre. Then

$$R(F, K_F) = k[X_0, X_1, Y, Z] / (Q_2, Q_6)$$

where $k[X_0, X_1, Y, Z]$ is the weighted polynomial ring with $\deg X_0 = \deg X_1 = 1$, $\deg Y = 2$, $\deg Z = 3$ and Q_2, Q_6 are homogeneous polynomials of degree 2 and 6 respectively of the form

$$Q_2 : \lambda Y - F_2(X_0, X_1)$$
$$Q_6 : Z^2 - Y^3 - F_6(X_0, X_1, Y).$$

Furthermore $\lambda \neq 0$ if and only if F is 2-connected. If $\lambda = 0$ then F_2 can be written as

$$X_0{}^2 - \alpha X_0X_1$$

with $\alpha \neq 0$ if F contains 2 distinct elliptic tails and $\alpha = 0$ if F contains one elliptic tail appearing with multiplicity 2 in F.

Proof The statement results from combining (3.7) and (3.8) below.

Remark 1 Note that, as a consequence of this theorem, we can write down a non-singular rational parameter space containing every fibre of genus 2.

Remark 2 If F is a fibre with $p_a(F) = 2$ then F is 1-connected, since $KF = 2$. If F is 2-connected the ring $R(F, K_F)$ is described in (I.9.4). If F is not 2-connected then F contains an elliptic tail (for properties of elliptic tails see section 2). If this happens we study $R(F, K_F)$ using the following division in two cases.

The following propositions will lead to the proof of (3.7).

(3.2) Proposition Let F be a fibre with $p_a(F) = 2$ containing an elliptic tail E. Then F can be decomposed as either:

Type (i) $F = E + E' + A$ where E and E' are two elliptic tails without common components and either $A = 0$, and $E.E' = 1$ or E and E' are disjoint and A is a simple chain of (-2)-curves linking E and E'

or

Type (ii) $F = 2E + B + \theta_0$, where B is a (-2)-cycle with $BE = 1$, θ_0 is a (-2)-curve satisfying $\theta_0 B = 0$ and $\theta_0 E = 1$.

Proof Let E be an elliptic tail in F. By (I.2.3) either $E \subset F-E$ or E and $F-E$ have no common components.

If E and $F-E$ have no common components then F can be decomposed as in (i). In fact let E' be the elliptic tail contained in the elliptic cycle $F-E$. Since $E' \neq E$, E' has no common components with $F-E'$ and either $F-E' = E$ in which case $EE' = 1$ or $A = F-(E+E') \neq 0$. So A has no common components with either E or E' and by the numerical properties of F (see I.8), $A^2 = -2$, $AE = 1$, $AE' = 1$. So A is a (-2)-cycle and it is easy to see that it must be of the form $\mathbf{A}_n$ (I.1.8).

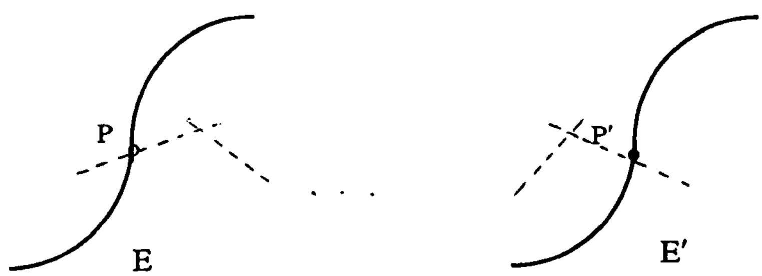
If $E \subset F-E$ then $F = 2E + A$ where $A^2 = -4$ and A is made up of (-2)-curves. Let $B \subset A$ be maximal subject to $B^2 = -2$ and let $C = A-B$. For each (-2)-curve θ in C, we have $\theta B \leq 0$ by maximality, hence $BC \leq 0$. Since

$-4 = B^2 + 2BC + C^2$ necessarily $BC = 0$ and $C^2 = -2$. From $BF = CF = 0$ we then get $BE = CE = 1$.

Since E is 2-connected $\theta E \geq 0$ for every (-2)-curve θ in A. Then let θ_0 be the only component of C such that $\theta_0 E = 1$. From $\theta_0 F = 0$ we have $\theta_0 C = -2$ and since C is a (-2)-cycle, $C = \theta_0$ and the result follows.

(3.3) **Remark**

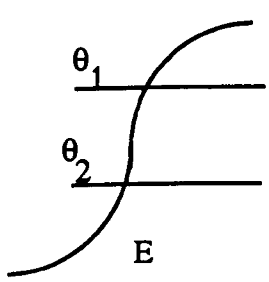
(1) If F has a decomposition as in (i) then F looks like



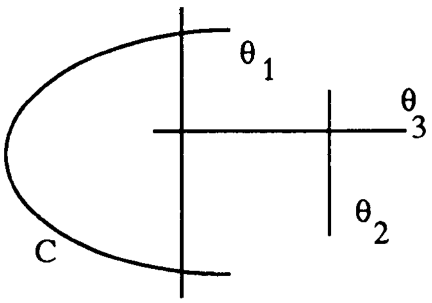
where E, E' are elliptic tails and the intersections are transversal.

If $P = E \cap (F-E)$ and $P' = E' \cap (F-E')$ then $\mathcal{O}_E(K_F) \cong \mathcal{O}_E(P)$ and $\mathcal{O}_{E'}(K_F) \cong \mathcal{O}_{E'}(P')$.

(2) If F has a decomposition as in (ii) F can be quite complicated due to common components between E and B. As an example we can have



where $F=2E+\theta_1+\theta_2$ or



where $\theta_1, \theta_2, \theta_3$ are (-2)-curves, $C^2 = -3$, $KC = 1$ and $E = C + \theta_1$, $F = 2C + 3\theta_1 + 2\theta_3 + \theta_2$.

(3.4) **Lemma** Let F be a fibre as in (3.2) and write $\Gamma = E + A$ if $F = E + E' + A$ (Type (i)) or $\Gamma = E + \theta_0$ if $F = 2E + B + \theta_0$ (Type (ii)). Let

$$f_n : H^0(F, nK_F) \to H^0(F-E, nK_F) \quad \text{and}$$
$$g_n : H^0(F, nK_F) \to H^0(F-\Gamma, nK_F)$$

be the restriction maps. Then

(i) f_n and g_n are surjective for every $n \geq 1$;

(ii) $\operatorname{Ker} f_n = \operatorname{Ker} g_n$ for every $n \geq 1$.

(iii) $\operatorname{Ker} g_1$ is generated as a k-vector space by a single element x_0 such that $x_{0|F-E} \equiv 0$ and the map $\operatorname{Ker} g_1 \otimes H^0(F, nK_F) \to \operatorname{Ker} g_{n+1}$ is surjective for every $n \geq 1$.

Proof (i) Since $\Gamma^2 = -1$, $E^2 = -1$, $K\Gamma = KE = 1$ both Γ and E are 1-connected with K nef and so (i) follows from proposition (I.6.9).

(ii) $\operatorname{Ker} f_n = H^0(E, nK_F-(F-E))$ and $\operatorname{Ker} g_n = H^0(\Gamma, nK_F-(F-\Gamma))$. Suppose that $\Gamma - E \neq 0$ otherwise there is nothing to prove; we have exact sequences

$$0 \to H^0\big(E, nK_F-(F-E)\big) \xrightarrow{\alpha_n} H^0\big(\Gamma, nK_F-(F-\Gamma)\big) \to H^0\big(\Gamma-E, nK_F-(F-\Gamma)\big).$$

Now $\Gamma - E$ is 1-connected and $\deg_{\Gamma-E}\big(nK_F-(F-\Gamma)\big) = -1$. Also by our choice of Γ, $\deg_\theta\left(nK_F - (F-\Gamma)\right) \le 0$ for each (-2)-curve θ in $\Gamma-E$, so by (I.3.1) $H^0\left(\Gamma-E, nK_F-(F-\Gamma)\right) = 0$. Therefore α_n is an isomorphism.

(iii) By (ii) $\operatorname{Ker} g_1 = \operatorname{Ker} f_1$. Since $\operatorname{Ker} f_1 = H^0(E,\mathcal{O}_E)$ it follows that $\operatorname{Ker} g_1$ is generated by a single element s such that $s_{|F-E} = 0$. Also by (ii) $\operatorname{Ker} g_n = \operatorname{Ker} f_n$, for every $n \ge 1$ and so it will be enough to prove (iii) for $\operatorname{Ker} f$.

The map $\operatorname{Ker} f_1 \otimes H^0(F, nK_F) \longrightarrow \operatorname{Ker} f_{n+1} = H^0(E, nK_F)$ factors through $\operatorname{Ker} f_1 \otimes H^0(F, nK_F) \longrightarrow \operatorname{Ker} f_1 \otimes H^0(E, nK_F) \longrightarrow \operatorname{Ker} f_{n+1}$ and thus (iii) is clear since $\operatorname{Ker} f_1 = H^0(E,\mathcal{O}_E)$ and the restriction maps

$$H^0(F, nK_F) \to H^0(E, nK_F)$$

are surjective.

(3.5) Lemma If $F = 2E + B + \theta_0$ is as in (3.2)(ii) and $\Gamma = E + \theta_0$ then the restriction maps

$$\beta_n : H^0(B + E, nK_F) \to H^0(E, nK_F)$$

are isomorphisms for every $n \ge 1$, and thus if $S' = R(B+E, nK_F)$, and $S = R(E, nK_F)$, the morphism $f: S' \to S$ induced by the restriction maps is an isomorphism.

Proof $\operatorname{Ker} \beta_n = H^0(B, nK_F-E)$. Since $\deg_B (nK_F-E) = -1$, B is 1-connected and $\theta(nK_F) = 0$, $\theta E \ge 0$ for every (-2)-curve θ in B, we can apply (I.3.1) and obtain $\operatorname{Ker} \beta_n = 0$. Hence β is an isomorphism by a dimension count.

(3.6) Proposition Let F, Γ be as in (3.4) and E the elliptic tail contained in Γ, E' the elliptic tail contained in $F-\Gamma$. Let $h : R = R(F, K_F) \to R(E', nK_F)$ be the morphism induced by the restriction maps. Then

(i) h is surjective.

(ii) Ker h is a principal ideal of $R(F, K_F)$ generated by an element x_0 of deg 1 in Ker h such that $x_{0|F-E} = 0$.

(iii) The ideal $J = \{x \in R : x_0 x = 0\}$ is the kernel of the surjective morphism $R(F, K_F) \to R(E, K_F)$ and is generated by its homogeneous elements of deg < 6.

Proof (i) This follows immediately from (3.4) and (3.5).

(ii) Let $R_n = H^0(F, nK_F)$. By (3.4) and (3.5) Ker $h \cap R_n$ = Ker f_n where $f_n : H^0(F, nK_F) \to H^0(F-E, nK_F)$ is the restriction map. Thus (ii) follows from (3.4).

(iii) The ideal J is the kernel of the linear map of graded algebras $R \to R$ given by $x \to x_0 x$ and thus it is an homogeneous ideal. But $J \cap R_n$ = Ker $\left(H^0(F, nK_F) \to H^0(E, nK_F)\right) = H^0(F-E, nK_F-E)$. Since $H^0(F, 2K_F)$ maps onto $H^0(F-E, 2K_F)$, $2K_F$ is free, and the map $H^0(F-E, nK_F-E) \otimes H^0(F, 2K_F) \to H^0(F-E, (n+2)K_F-E)$ factors through $H^0(F-E, 2K_F)$, using Castelnuovo's lemma, we get that the map above is surjective for $n \geq 4$. Hence $J_n R_2 = J_{n+2}$, for $n \geq 4$.

(3.7) Theorem Let F be a fibre with $p_a(F) = 2$ containing an elliptic tail E. Then

$$R(F,K_F) = k[X_0,X_1,Y,Z] / I$$

where $k[X_0,X_1,Y,Z]$ is the weighted polynomial ring with $\deg X_0 = \deg X_1 = 1$, $\deg Y = 2$ and $\deg Z = 3$ and $I = (Q_2, Q_6)$ with Q_2 a non zero homogeneous polynomial of degree 2 in X_0, X_1 and Q_6 a homogeneous polynomial of degree 6 in which both Z^2 and Y^3 appear with non-zero coefficient.

Furthermore (Q_2, Q_6) can be chosen such that

$$Q_2 = X_0^2 - \lambda X_0 X_1$$

$$Q_6 = Z^2 - Y^3 - X_1^2(\alpha_0 Y^2 + \alpha_1 X_1^4) - X_0 X_1 (\beta_0 Y^2 + \beta_1 X_1^4),$$

where $\lambda \neq 0$ if F contains 2 distinct elliptic tails and $\lambda = 0$ if F contains one elliptic tail appearing with multiplicity 2.

Proof Let Γ be as in (3.4) and $E' \subset F-\Gamma$, $E \subset F-E'$ the elliptic tails of F. Let $R = R(F, K_F)$ and $S = R(E', K_F)$. By (3.5) the morphism $h : R \to S$ induced by the restriction maps is surjective and its kernel is the principal ideal generated by $x_0 \in R_1$ with $x_{0|F-E} = 0$. The ideal $J = \{x \in R : x_0 x = 0\}$ is the kernel of the surjective morphism $R \to R(E, K_F)$ and is generated by its homogeneous elements of deg 6.

Both E' and E are 2-connected genus 1 divisors and K_F has degree 1 on E' and E. In (I 7.3) we have seen that if L is a sheaf of degree 1 on a 2-connected genus 1 divisor $R(D, L) = k[X, Y, Z] / (F_6)$ where $\deg X = 1$, $\deg Y = 2$, $\deg Z = 3$ and F_6 is a polynomial of degree 6 in which both Z^2 and Y^3 appear with non-zero coefficient. In particular $R(E, K_F)$ is of this form and thus we can apply the general theory outlined in section 1 (J satisfies the conditions stated in (1.6)).

Let $S = k[X_1, Y, Z] / I$ where I is generated by F_6 as above. By section 1

$$R = k[X_0, X_1, Y_1, Z] / \tilde{I}$$

where $x_0 \in R_1$ is such that $x_{0|F-E} = 0$. Since y and z are the only new generators in degree 2 and 3 of R, necessarily $R(E, K_F)$ will be generated by the restrictions of y, z and some linear combination of x_0 and x_1, and the ideal J

will be generated by a single element x such that $x_{|E} = 0$.

If $E = E'$ then $x_{0|E} = 0$ and thus $\langle x_{1|E'} \rangle = H^0(E, K_F)$, and J is generated by x_0.

If $E \neq E'$, since $x_{0|F-E} = 0$, then $x_{0|E} \neq 0$ and thus $x_{1|E} = \alpha x_{0|E}$ i.e. J is generated by $x_1 - \alpha x_0$. Since adding a scalar multiple of x_0 to x_1 will not affect anything in S or I we can assume that $\alpha \neq 0$ and thus $\langle x_{1|E} \rangle = H^0(E, K_F)$ and J is generated by $x_0 - \lambda x_1$, $\lambda \neq 0$.

Then (as in section 1 with $x_i = x_1$, $x_j = y$, $x_k = z$) $(x_0) \subset R$ is such that $(x_0) = x_0 \ (k[x_1, y] + k[x_1, y])$ and is isomorphic as a k-vector space to $X_0(k[X_1, Y] + Z\, k[X_1, Y])$.

So $\tilde{I} = \left(X_0^2 - \lambda X_0 X_1 , \tilde{F}_6\right)$

where $\tilde{F} = F_6 - X_0 P_5(X_1, Y, Z)$, and $\lambda \neq 0$ if $E \neq E'$, $\lambda = 0$ if $E = E'$.

With respect to the form of $\tilde{F}_6$ note that F_6 could have been chosen to be

$Z^2 - Y^3 - \alpha_0 X_1^4 Y - \alpha_1 X_1^6$ and thus $\tilde{F}_6$ would be

$$Z^2 - Y^3 - \alpha_0 X_1^4 Y - \alpha_1 X_1^6 - \gamma_0 X_0 X_1 Y^2 - \gamma_1 X_0 X_1^3 Y - \gamma_2 X_0 X_1^5 - \gamma_3 X_0 X_1^2 Z - \gamma_4 X_0 Y Z .$$

Since a change of coordinates of Y, Z by a multiple of X_0 does not affect S or F_6 it is possible to change Y, Z so that $\tilde{F}_6$ is

$$Z^2 - Y^3 - \alpha_0 X_1^4 Y - \alpha_1 X_1^6 - \beta_0 X_0 X_1^3 Y - \beta_1 X_0 X_1^5 .$$

Notice that $\tilde{F}_{6|E}$ is the equation describing E.

(3.8) Proposition Let F be a 2-connected fibre with $p_a(F) = 2$. Then $R(F, K_F) = k[X_0, X_1, Z] / (F_6)$, where $k[X_0, X_1, Z]$ is the weighted polynomial ring with $\deg X_0 = \deg X_1 = 1$, $\deg Z = 3$ and F_6 is an homogeneous polynomial of deg 6 in which Z^2 appears with non zero coefficient.

Proof This is done in (I.9.4)

Section 4. Numerical decomposition of a genus 3 fibre.

(4.1) In section 3 we gave a complete description of $R(F, K_F)$ for a genus 2 fibre. For $p_a(F) = 3$, $R(F, K_F)$ is much more complicated and the rest of this chapter sets up the ground for the complete description of $R(F, K_F)$. That will be done in Chapter III. In this section we establish a decomposition of F for a 1-connected fibre F of genus 3 similar to the one established in (3.2) for F a fibre of genus 2, and we describe in detail this decomposition in (4.3) to (4.16).

This decomposition of F provides a first purely numerical division of F into cases; note that this is not the main division used in Chapter III.

(4.2) **Theorem** Let F be a 1-connected fibre with $p_a(F) = 3$. Then:

There exists a decomposition of $F = \sum_{i=1}^{n} A_i + D$ with $0 \leq n \leq 3$ such that each A_i is an elliptic cycle, D is a 2-connected divisor and:

(i) $A_i A_j = 0$, for $i \neq j$.

(ii) $i < j \Rightarrow A_j \subset A_i$ or A_i and A_j are disjoint;

(iii) For $k \leq j$ and for every (-2)-curve θ contained in A_j we have

$$\theta\Big(\sum_{i=1}^{k} A_i\Big) \leq 0 .$$

(iv) If E_j is the elliptic tail contained in A_j, for every $1 \leq i < j \leq n$ and for every (-2)-curve θ in $A_j - E_j$, $\theta A_i = 0$.

(v) If $A_j - E_j \neq 0$ then $E_j \left(\sum_{k>j} A_k + D\right) = 0$.

(vi) Furthermore if $n = 3$, then $D \cong \mathbb{P}^1$ and D is not contained in $\mathrm{Supp}\left(\sum_{i=1}^{3} A_i\right)$.

(4.3) Definition A decomposition satisfying the conditions in (a) of theorem (4.2) is a *standard numerical decomposition* of F.

Proof of (4.2) Remark that (v) is a trivial consequence of (iv) by the properties of elliptic cycles (see section 2).

If F is 2-connected there is nothing to prove. If F is not 2-connected then F contains an elliptic cycle. Let $A_1 \subset F$ be a maximal elliptic cycle. Then $B_1 = F - A_1$ is 1-connected (by I.2.1), and for every (-2)-curve θ in F, $\theta A_1 \leq 0$ by the maximality of A_1.

If B_1 is 2-connected we get the required decomposition. Now the remainder of the proof is similar.

If B_1 is not 2-connected then $B_1 = A+B$ with $AB = 1$. Since $B_1^2 = -1$ we have $A^2 + B^2 = -3$. By maximality of A_1 subject to $A_1(F-A_1) = 1$, $KA_1 = 1$, we must have $KA > 0$, $KB > 0$. Since for any divisor Γ $K\Gamma \equiv \Gamma^2 \bmod 2$ and $KB_1 = 3$, either A or B is an elliptic cycle.

Let $A_2 \subset B_1$ be maximal subject to $KA_2 = 1$, $A_2(B_1 - A_2) = 1$ and let $B_2 = B_1 - A_2$. Since $A_2^2 = -1$ and $A_2F = 0$, $A_2A_1 = 0$. By (2.9), either

A_1 and A_2 are disjoint or one of them contains the other. By the maximality of A_1 either A_1 and A_2 have no common components or $A_2 \subset A_1$ (thus (ii) is satisfied for $i,j \leq 2$). Now we are going to prove (iii) and (iv) for $i,j \leq 2$.

For every (-2)-curve θ in B_1, $\theta A_1 \leq 0$, and $\theta A_2 \leq 1$. If $\theta(A_1+A_2) \geq 1$ then $\theta A_1 = 0$, $\theta A_2 = 1$ and $\theta \subset B_2$. But this contradicts the maximality of A_2 with respect to $A_2(B_1-A_2) = 1$, $KA_2 = 1$. Hence for every (-2)-curve θ in B_1, $\theta(A_1+A_2) \leq 0$, and so (iii) is satisfied.

If A_1 and A_2 are disjoint (iv) is trivially satisfied. Otherwise let E be the elliptic tail contained in A_2. By (2.5), $E \subset A_1$ and since $A_1 - E \neq 0$, $EA_1 = 0$. From $A_2A_1 = 0$ we have $(A_2-E)A_1 = 0$. Since for every (-2)-curve θ in A_2, $\theta A_1 \leq 0$ necessarily $\theta A_1 = 0$ for every $\theta \in A_2-E$ so (iv) is satisfied.

If B_2 is 2-connected we again have the required decomposition with $n = 2$.

Otherwise B_2 decomposes as $B_2 = A+B$ with $AB = 1$. Again we must have $KA > 0$, $KB > 0$, and $A^2 = -1$ or $B^2 = -1$. Let $A_3 \subset B_2$ be maximal with respect to $A_3{}^2 = -1$, $A_3(B_2-A_3) = 1$ and let $B_3 = B_2-A_3$. From $A_3F = 0$ we get $A_3(A_1+A_2) = 0$. By (2.7) $A_3A_1 \leq 0$, $A_3A_2 \leq 0$ and thus $A_3A_1 = A_3A_2 = 0$. Now (ii) follows trivially from (2.9).

Since for every (-2)-curve θ in B_1, $\theta(A_1 + A_2) \leq 0$ using again the maximality of A_3 we have $\theta(A_1+A_2+A_3) \leq 0$ for every (-2)-curve θ in B_2, and so (iii) is satisfied.

If A_3 is disjoint from A_2 and A_1, then (iv) is trivially satisfied. Otherwise, if E is the elliptic tail contained in A_3, since $EA_1 = EA_2 = 0$, then $(A_3-E)A_1 = (A_3-E)A_2 = 0$ and hence for every θ in A_3-E, $\theta A_1 = \theta A_2 = 0$. So (iv) is satisfied.

Now B_3 is 1-connected and $B_3^2 = -3$, $KB_3 = 1$, hence $p_a(B_3) = 0$. If B_3 is 2-connected then B_3 is necessarily isomorphic to $\mathbb{P}^1$. Suppose that B_3 is not 2-connected. Then B_3 would contain A such that $A^2 = -2$, $KA = 0$, $A(B_3-A) = 1$ and this contradicts the way in which A_1, A_2, A_3 were chosen.

So B_3 is irreducible. If B_3 is contained in A_i, then B_3 is the only component C_0 of E_i such that $C_0E_i = -1$, and $A_i-E_i \neq \emptyset$. From $B_3A_i = 1$ we would have $B_3(A_i-E_i) = 2$. Hence $(B_3+A_i-E_i)^2 = -1$ and so A_i would contain two distinct elliptic tails contradicting (2.5). Thus $B_3 \not\subset \mathrm{Supp}(\Sigma A_i)$.

(4.4) **Proposition** If F is as in (4.2) and has a standard decomposition $F = A_1 + D$ with $n = 1$ then either

(i) A_1 and D have no common components and if E_1 is the elliptic tail contained in A_1 and $A_1-E_1 \neq \emptyset$ then A_1-E_1 and E_1 have also no common components

or

(ii) $A_1 \subset D$ and $A_1 = E$ is an elliptic tail appearing with multiplicity one in D. Furthermore, E is the only elliptic tail contained in D.

Proof If $A_1 \subset D$ we are going to prove that A_1 is 2-connected and thus by (2.4) A_1 is an elliptic tail. If A_1 were not 2-connected, we would have a decomposition of A_1, $A_1 = J_1 + J_2$ with $J_1J_2 = 1$ and say $J_1^2 = -1$. Then $J_1A_1 = 0$ and so (from $J_1F = 0$) $J_1D = 0$, contradicting 2-connectedness of D.

By 2-connectedness of D it is clear that $E = A_1$ appears with multiplicity 1 in D. Furthermore, if D contains an elliptic tail E_1, by 2-connectedness of D we have $E_1D > 0$, and thus by (2.5) $E_1 = E$.

Now suppose that $A_1 \not\subset D$. Write

$A_1 = M + B$

$D \;= M + C$

where B and C have no common components. Then $B \neq 0$ by assumption and C is trivially different from 0. Then

$$0 \leq BC = (A_1-M)(D-M) = A_1D - FM + M^2 = 1 + M^2 .$$

If $M \neq 0$ then $M^2 < 0$, and therefore $M^2 = -1$. But this is impossible since by 1-connectedness of A_1 and 2-connectedness of D, $MB \geq 1$, $MC \geq 2$ and thus $M^2 \leq -3$.

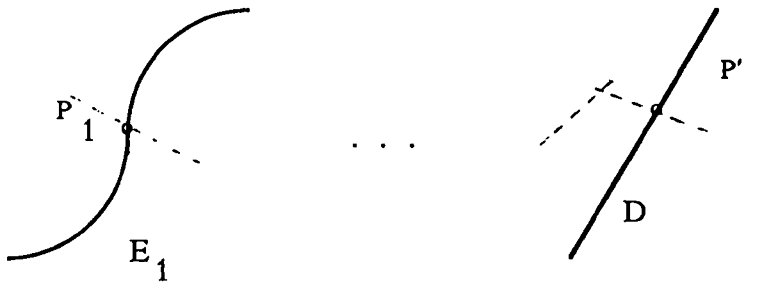
So either $A_1 \subset D$ or A_1 and D have no common components.

By (2.7) either A_1-E_1 and E_1 have no common components or $A_1-E_1 = \theta_0 + B$ where $\theta_0E_1 = 1$, $\theta_0B = 1$, $B \subset E_1$ and $BE_1 = 0$. But then it is clear that if A_1 and D have no common components, then E_1 and A_1-E_1 have no common components (otherwise $BD > 0$, $E_1D > 0$, $\theta_0D = 0$, contradicting $D(-D) = 1$).

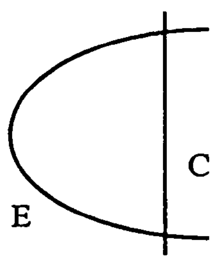
(4.5) Corollary If F is as in (4.4.(i)) then A_1 and D intersect transversally in a non-singular point P of D, and $\mathcal{O}_D(K_F) \cong \mathcal{O}_D(K_D+P)$. Also, if E_1 is the elliptic tail contained in A_1, $\mathcal{O}_{E_1}(K_F) \cong \mathcal{O}_{E_1}(P_1)$ where $P_1 = E_1 \cap (F-E_1)$. If $A_1-E_1 \neq \emptyset$, then A_1-E_1 is a chain of (-2)-curves linking E_1 to D.

Proof The statement above comes trivially from (4.4.(i)) and the fact that for any divisor $D' \subset F$, $\mathcal{O}_{D'}(K_F) = \mathcal{O}_{D'}(K+D'+F-D')$.

(4.6) Remarks So if F has a standard decomposition with $n = 1$, either F in case (i) looks like:

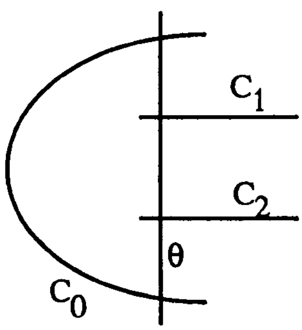


or F in case (ii) can be quite complicated. Numerical examples for case (ii) can be, for instance



with $F = 2E + C$, E an elliptic tail, C a rational curve with $C^2 = -4$,

or



with $E = C_0 + \theta$, $F = 2\,C_0 + 3\theta + C_1 + C_2$, where θ is a (-2)-curve and C_0, C_1, C_2 are rational curves with $C_i^2 = -3$.

(4.7) **Proposition** If F is as in (4.2) and has a standard decomposition $F = A_1 + A_2 + D$ with $n = 2$, then

(i) A_2 and D have no common components and D contains no elliptic tail.

(ii) If E_2 is the elliptic tail contained in A_2 then E_2 and A_2-E_2 have no common components.

(iii) If A_1 and A_2 are disjoint then (i) and (ii) are also true for A_1.

Proof Since D is 2-connected, $A_2 \not\subset D$. Otherwise from $A_2 D = 1$ we would have $A_2 (D-A_2) = 2$ and $(D-A_2)^2 = -5$. Thus $(D-A_2)$ would not be 1-connected but $(D-A_2) A_2 = 2$, and this contradicts lemma (I.2.2).

The same argument shows easily that D does not contain any elliptic tail.

Suppose now that A_2 has common components with D. We can write:

$$A_2 = A + B$$
$$D = A + C$$

where $A, B, C \neq 0$ and B and C have no common components.

Then

$$A^2 + AB + AC + BC = 1 \quad (\text{from } A_2D = 1)$$
$$2A^2 + AA_1 + AB + AC = 0 \quad (\text{from } A_2F = 0) \text{ and thus}$$
$$A^2 + AA_1 = -1 + BC.$$

Since $BC \geq 0$, $AB \geq 1$, $AC \geq 2$, we have $A^2 \leq -2$ and thus $AA_1 > 0$. From $(A_1+A)^2 < 0$ we get that $AA_1 \leq 1$ and so $AA_1 = 1$ and $A^2 = -2$. But A is made up of (-2)-curves and by the maximality of A_1, $AA_1 \leq 0$ giving a contradiction. Thus A_2 and D have no common components.

Now (ii) follows from the description of A_2 given in (2.7), as in Proposition (4.4) and (iii) is also clear.

(4.8) Corollary If F is as in (4.2) and has a decomposition $F = A_1+A_2+D$ with $n = 2$ and A_1 and A_2 disjoint, then A_i and D intersect transversally in distinct nonsingular points P_i of D (for $i = 1,2$) and $\mathcal{O}_D(K_F) \cong \mathcal{O}_D(P_1 + P_2)$. Also if E_i is the elliptic tail contained in A_i, then $\mathcal{O}_{E_i}(K_F) \cong \mathcal{O}_{E_i}(Q_i)$ where $Q_i = E_i \cap F-E_i$ and if $A_i-E_i \neq \emptyset$ then A_i-E_i is a chain of (-2)-curves linking E_i to A_i-E_i.

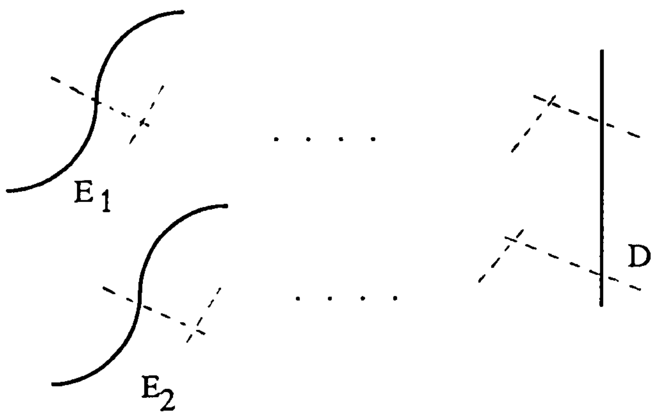
Proof The statement is obvious by proposition (4.7), the fact that $\mathcal{O}_{D'}(K_F) = \mathcal{O}_{D'}(K + D' + F-D')$ for any divisor contained in F, and that if A_1 and A_2 are disjoint then $\mathcal{O}_{E_i}(A_j) \cong \mathcal{O}_{E_1}$.

(4.9) Corollary If F is as in (4.2) and has a standard decomposition $F = A_1 + A_2 + D$ with $n = 2$ and $A_1 \supset A_2$ then A_2 intersects D_2 transversally in a non-singular point P of D_2, E_2 intersects $(A_2 + D_2 - E_2)$ transversally in a non-singular point Q of E_2. Furthermore $\mathcal{O}_{E_2}(K_F) \cong \mathcal{O}_{E_2}(Q)$ if and only if $\mathcal{O}_{E_2}(A_1) \cong \mathcal{O}_{E_2}$.

Proof The statement follows immediately from the proposition above. That $\mathcal{O}_{E_2}(K_F) \cong \mathcal{O}_{E_2}(Q)$ if and only if $\mathcal{O}_{E_2}(A_1) \cong \mathcal{O}_{E_2}$ is clear from the fact that

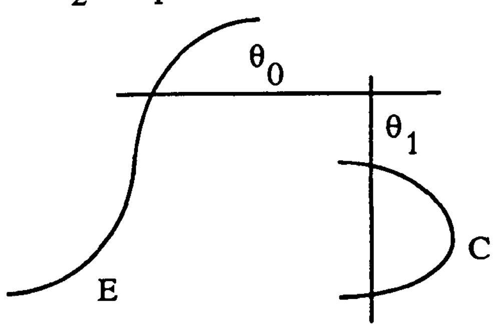
$$\mathcal{O}_{E_2}(K_F) \cong \mathcal{O}_{E_2}(K+F) \cong \mathcal{O}_{E_2}(F-E_2) \cong \mathcal{O}_{E_2}(A_1+Q).$$

(4.10) **Remark** By the statements above, if F has a standard decomposition with $n = 2$, either A_1 and A_2 are disjoint and F looks like



where the dotted chains may be 0.

or $A_2 \subset A_1$ and F can be quite complicated. A numerical example can be



where θ_0, θ_1 are -2-curves, C is a rational curve with self intersection -4,
$F = 2E + 2\theta_0 + 2\theta_1 + C$ and $A_1 = E + \theta_0 + \theta_1$, $A_2 = E + \theta_0$, $D = \theta_1 + C$.

(4.11) **Proposition** If F is as in (4.2) and has a standard decomposition $F = A_1 + A_2 + A_3 + D$ with $n = 3$ then

(i) A_3 and D intersect transversally in a point P, and if E_3 is the elliptic tail contained in A_3, E_3 and $A_3 - E_3$ have no common components.

(ii) If A_2 and A_3 are disjoint, (i) is also true for A_2.

(iii) If A_1 is disjoint from A_3 and A_2, (i) is also true for A_1.

Proof The proof is similar to the proofs of (3.4), (3.7) and we omit it.

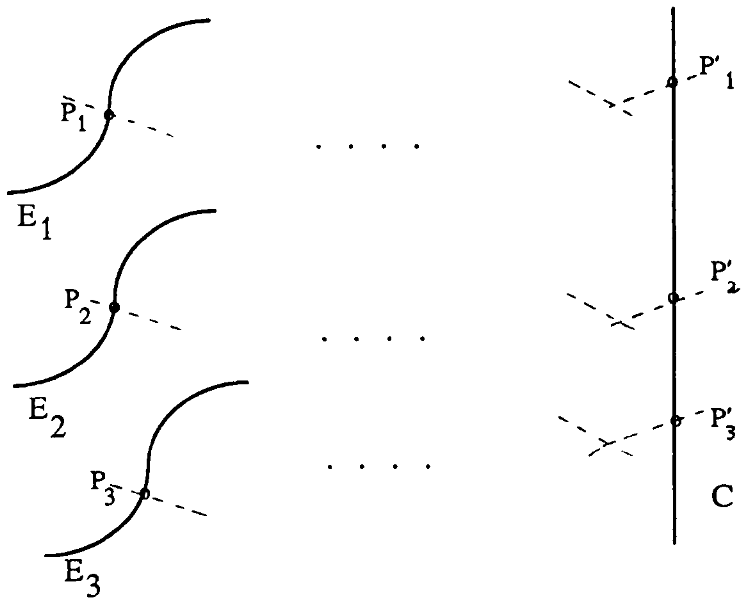
(4.12) **Corollary** If F is as in (4.2) and has a standard decomposition with $n = 3$ and A_i is disjoint from $A_k + A_j$ ($\{i,j,k\} = \{1, 2, 3\}$) then $A_i + D$ is the elliptic tail $E_i \subset A_i$ linked to D by a simple chain of (-2)-curves and $\mathcal{O}_{E_i}(K_F) = \mathcal{O}_{E_i}(Q_i)$ where $Q_i = E_i \cap (A_i + D - E_i)$.

(4.13) **Corollary** If F is as in (4.2) and has a decomposition with $n = 3$ and $A_k \subset A_j$, A_i and A_j disjoint ($\{i,j,k\} = \{1,2,3\}$) then $A_k + D$ and $A_i + D$ are the elliptic tails E_k and E_i respectively linked to D by a simple chain of (-2)-curves, (possibly 0). In these conditions $\mathcal{O}_{E_i}(K_F) \cong \mathcal{O}_{E_i}(Q_i)$ where $Q_i = E_i \cap (F-E_i)$. If $Q_k = E_k \cap (A_K + D - E_K)$, then $\mathcal{O}_{E_k}(Q_k) \cong \mathcal{O}_{E_k}(K_F)$ if and only if $\mathcal{O}_{E_k}(A_j) \cong \mathcal{O}_{E_k}$.

(4.14) **Corollary** If F (F as in (4.2)) has a decomposition with $n = 3$ and $A_3 \subset A_2 \subset A_1$, then A_3+D is just the elliptic tail E_3 linked to D by a simple chain of (-2)-curves (possibly 0). If $Q_3 = E_3 \cap (D+A_3-E_3)$, then $\mathcal{O}_{E_3}(Q_3) \cong \mathcal{O}_{E_3}(K_F)$ if and only if $\mathcal{O}_{E_3}(A_1+A_2) \cong \mathcal{O}_{E_3}$.

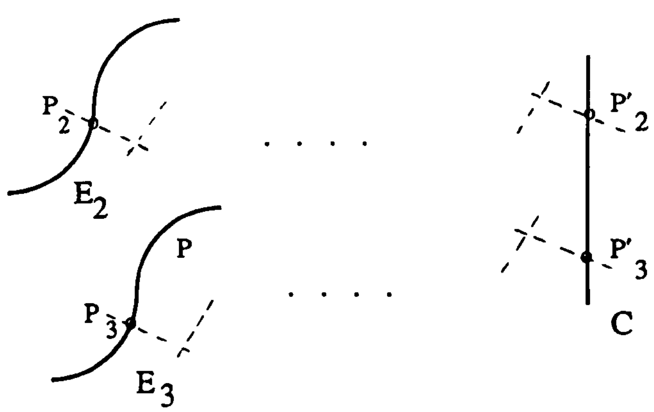
(4.15) **Remarks**

(1) If F has a standard decomposition with $n = 3$ and all the A_i's disjoint, then F looks like

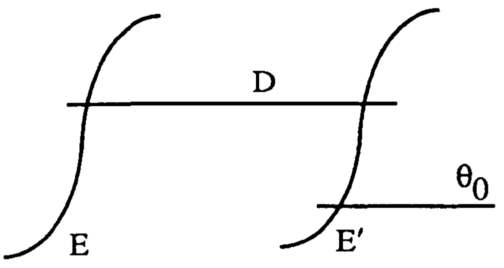


where $C \cong \mathbb{P}^1$.

(2) If F has a standard decomposition with A_1 disjoint from A_3, $A_1 \supset A_2$ then F-A_1 looks like

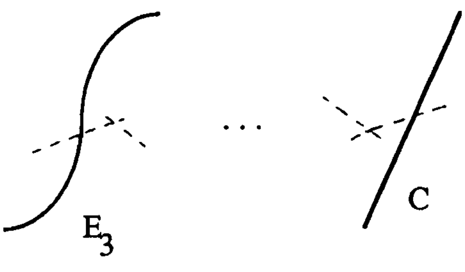


A numerical example of this case can be

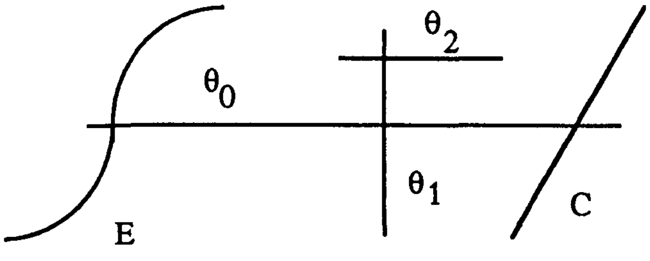


where θ_0 is a (-2)-curve, E and E′ are elliptic tails, KD = 1 , $D^2 = -3$ and $F = E + 2E' + \theta_0 + D$, $A_1 = E$, $A_2 = E' + \theta_0$, $A_3 = E'$.

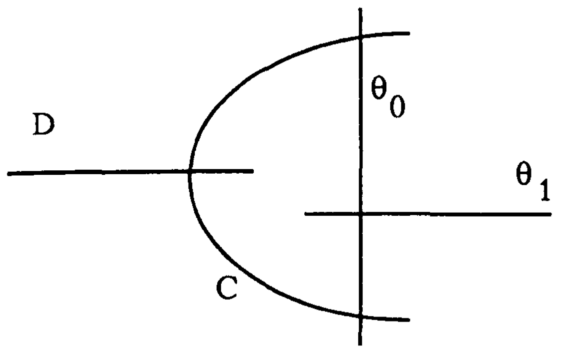
(3) If F has a standard decomposition with $A_1 \supset A_2 \supset A_3$, then $A_3 + C$ looks like



Numerical examples of this case can be,



where E is an irreducible elliptic tail, θ_0 , θ_1 are (-2)-curves , KC = 1 , $C^2 = -3$ and $F = 3E + 3\theta_0 + 2\theta_1 + \theta_2 + C$, $A_1 = E + \theta_0 + \theta_1 + \theta_2$, $A_2 = E + \theta_0 + \theta_1$, $A_3 = E + \theta_0$, or



where θ_0 and θ_1 are (-2)-curves, C and D are rational curves with $C^2 = -3$, $D^2 = -3$, and $F = 3C + 4\theta_0 + 2\theta_1 + D$, $E = C + \theta_0$, $A_1 = C + 2\theta_0 + \theta_1$, $A_2 = C + \theta_0 + \theta_1$, $A_3 = C + \theta_0$.

(4.16) **Remark** As a consequence of the above propositions it is clear that although F does not necessarily have a unique standard decomposition. The following happens:

(1) If a standard decomposition has $n = 1$ then it is unique.

(2) If a standard decomposition has $n = 2$ and $A_1 \supset A_2$ then the decomposition is the unique standard decomposition of F. If A_1 and A_2 are disjoint then F has another standard decomposition obtained by changing the order of A_1 and A_2.

(3) If a standard decomposition has $n = 3$ and $A_1 \supset A_2 \supset A_3$, then this decomposition is the unique standard decomposition of F. If A_k is disjoint from A_i, and $A_i \supset A_j$ ($\{i,j,k\} = \{1,2,3\}$, $i < j$), any other standard decomposition of F is obtained by a permutation σ of $\{1,2,3\}$ such that $\sigma(i) < \sigma(j)$. If A_i, A_j, A_k are pairwise disjoint ($\{i,j,k\} = \{1,2,3\}$) then any other standard decomposition of F can be obtained by a permutation of $\{1,2,3\}$.

Section 5. Properties of $R(F, K_F)$ for a 1-connected genus 3 fibre.

(5.1) In this section we show that if F is a 1-connected genus 3 fibre then $R(F, K_F)$ is generated by its elements of degree less than or equal to 3 (Theorem 5.5). We also prove some further properties of F that will be used in the calculations in Chapter III.

(5.2) Lemma Let $F = \sum_{i=1}^{n} A_i + D$ be a standard decomposition of F as in (4.2). For each i let E_i be the elliptic tail contained in A_i. Then:

(i) The restriction maps

$$H^0(A_i, nK_F) \to H^0(E_i, nK_F)$$

are isomorphisms for every $n \geq 0$.

(ii) The restriction maps

$$H^0(A_i, nK_F - \sum_{j<i} A_j) \to H^0(E_i, nK_F - \sum_{j<i} A_j)$$

are isomorphisms for every $n \geq 1$.

(iii) $H^0\left(A_i - E_i, nK_F - \left(\sum_{k>i} A_k + D\right)\right) = 0$, for every $n \geq 1$, and hence

$$H^0(E_i, (n-1)K_F + \sum_{j<i} A_j) = H^0\left(A_i, nK_F - (\sum_{k>i} A_k + D)\right).$$

Proof If $A_i - E_i = \emptyset$ there is nothing to prove. Otherwise we have seen in (2.7) that $A_i - E_i$ is a (-2)-cycle of type A_n such that $(A_i - E_i)E_i = 1$. Since

$$\mathrm{Ker}\left\{H^0(A_i, nK_F) \to H^0(E_i, nK_F)\right\} = H^0(A_i - E_i, nK_F - E_i),$$

$\deg(nK_F - E_i)_{|A_i - E_i} = -1$, and for every θ in $A_i - E_i$, $\theta E_i \geq 0$, $\deg(K_{F|\theta}) = 0$, the sheaf $(nK_F - E_i)$ does not have any sections on $A_i - E_i$ by (I.3.1). Hence (i) follows easily by a dimension count.

Now (ii) can be proved in the same way using the properties of standard decompositions.

For (iii) remark that since $\mathcal{O}_{E_i}(K+E_i) \cong \mathcal{O}_{E_i}$,

$$H^0(E_i, (n-1)K_F + \sum_{j<i} A_j) = H^0(E_i, nK_F - (\sum_{k>i} A_k + D + A_i - E_i)) =$$

$$= \mathrm{Ker}\left\{H^0(A_i, nK_F - (\sum_{k>i} A_k + D)) \longrightarrow H^0(A_i - E_i, nK_F - (\sum_{k>i} A_k + D))\right\}$$

and thus (iii) follows again from (I.3.1), because, by the properties of a standard decomposition $\theta(\sum_{k>i} A_k + D) \geq 0$,for every θ in $A_i - E_i$, and $(A_i - E_i)(\sum_{k>i} A_k + D) > 0$.

(5.3) Lemma Let $F = \sum_{i=1}^{\ell} A_i + D$ as in (4.2) be a standard decomposition of F and E_i the elliptic tail contained in A_i. Then

(i) $h^0(E_i, \sum_{j<i} A_j) \neq 0$ if and only if $\mathcal{O}_{E_i}(\sum_{j<i} A_j) \cong \mathcal{O}_{E_i}$.

(ii) $h^0(E_i, (n-1)K_F + \sum_{j<i} A_j) = n-1$, for $n \geq 2$.

(iii) If the standard decomposition of F is such that $\ell = 3$, and $A_1 \supset A_2 \supset A_3$, then $\mathcal{O}_E(A_1 + A_2) \not\cong \mathcal{O}_E$ implies that $\mathcal{O}_E(A_1) \not\cong \mathcal{O}_E$.

Proof (i) Assume that $h^0(E_i, \sum_{j<i} A_j) \neq 0$. Since $E(\sum_{j<i} A_j) = 0$ if E is irreducible (i) follows from (I.3.3).

Otherwise let $s \neq 0$, $s \in H^0(E_i, \sum_{j<i} A_j)$. Then (by I.3.1) either s does not vanish identically on any component Γ of E_i and thus by (I.3.3) $\mathcal{O}_{E_i}(\sum_{j<i} A_j) \cong \mathcal{O}_{E_i}$, or there exists a decomposition $E = D_1 + D_2$ such that $D_1 D_2 \leq D_1 (\sum_{j<i} A_j)$.

But for every (-2) curve θ in E_i, $\theta(\sum_{j<i} A_j) \leq 0$ by the properties of a standard decomposition and thus, if C_0 is the unique component of E_i such that $K_{C_0} = 1$, by 2-connectedness of E, $C_0(\sum_{j<i} A_j) \geq 2$. This is impossible. In fact, since $C_0 A_i \leq 1$, (because $(C_0+A_1)^2 < 0)$, $C_0(\sum_{j<i} A_j) = 2$, if and only if $i = 3$. But $C_0(E_i + \sum_{j<i} A_j) = 1$ hence $C_0(F-(A_1+A_2+E_i)) < 0$, which is a contradiction.

So (i) is proved.

(ii) follows trivially from Riemman Roch and

$$H^1(E_i, (n-1)K_F + \sum_{j<i} A_j) \cong H^0(E_i, (1-n)K_F - \sum_{j<i} A_j).$$

In fact we have seen in the proof of (i) that if C_0 is the only component of E_i such that $KC_0 = 1$, $C_0(\sum_{j<i} A_j) \leq 1$ and thus E_i contains at most one (-2)-curve θ such that $\theta(\sum_{j<i} A_j) = -1$.

Since for $n > 2$ $\deg_E((1-n)K_F - \sum_{j<i} A_j) < 0$ the result follows from 2-connectedness of E.

(iii) If E and A_1-E have no common components, the result is obvious since then $\mathcal{O}_E(A_1) \cong \mathcal{O}_E(A_2)$ ($A_2 \neq E$ because E appears with multiplicity 3).

If E and A_1-E have common components then by (2.7) $A_1 = B+\theta_0+E$ where B is made up of (-2)-curves in E and $\theta_0 \not\subset E$, $\theta_0 B = \theta_0 E = 1$. Since $BE = 0$, $BA_1 < 0$ and so there exists $\theta \in E$ such that $\theta A_1 < 0$. Thus $\mathcal{O}_E(A_1) \not\cong \mathcal{O}_E$.

(5.4) Proposition Let $F = \sum_{i=1}^{\ell} A_i + D$ be a standard decomposition of F as in 4.2 and

$$\varphi_{n,k} : H^0(F, nK_F) \to H^0(F - \sum_{i=1}^{k} A_i, nK_F)$$

be the restriction maps $(1 \leq k \leq \ell)$. Then

(i) $\varphi_{n,k}$ is surjective for every k and $n \geq 2$.

(ii) $\varphi_{1,1}$ is surjective.

(iii) The codimension of $\operatorname{Im} \varphi_{1,2}$ is equal to $h^0(E_2, \mathcal{O}_{E_2}(A_1))$ and the codimension of $\operatorname{Im} \varphi_{1,3}$ is equal to $h^0(E_2, \mathcal{O}_{E_2}(A_1)) + h^0(E_3, \mathcal{O}_{E_3}(A_1+A_2))$ (E_i is the elliptic tail contained in A_i).

Remark This proposition is the key proposition that gives the analytical decomposition used in (III.1).

Proof (i) For $n > 2$

$$\text{coker } \varphi_{n,k} = H^0\left(\sum_{i=1}^{k} A_i\,,\, nK_F - \left(F - \sum_{i=1}^{k} A_i\right)\right) \cong H^0\left(\sum_{i=1}^{k} A_i\,,\,(1-n)K_F\right)$$

and the result follows easily from the previous lemmas.

Now (ii) and (iii) follow easily from the fact that

$\text{Ker } \varphi_{1,k} = H^0\left(\sum_{i=1}^{k} A_i, K + \sum_{i=1}^{k} A_i\right)$ and (5.2)(iii), (5.3).

(5.5) Theorem If F is a 1-connected fibre with $p_a(F) = 3$ then the k-algebra $R(F, K_F) = \oplus H^0(F, nK_F)$ is generated by its elements of degree ≤ 3.

Proof Let us denote $H^0(F, nK_F)$ by R_n. By (I.9.3) we know that $R(F, K_F)$ is generated by its elements of $\deg \leq 5$. If F is 2-connected the result is proved in (I.9.9). Otherwise F admits a standard decomposition $F = \sum_{i=1}^{\ell} A_i + D$ with $\ell \geq 1$. Let E_i be the elliptic tail contained in A_i. The restriction maps φ_n from $H^0(F, nK_F) \to H^0(E_1, nK_F)$ are surjective (by I.6.9).

Thus the map $\varphi : R(F, K_F) \to R(E_1, K_F)$ is surjective. K_F is a sheaf of degree 1 on E_1 such that $\deg K_{F|_\Gamma} \geq 0$, for all $\Gamma \subset E_1$ and so (by I.7.3) $R(E_1, K_F)$ is generated by its elements of degree ≤ 3. $\text{Ker } \varphi = \oplus \text{Ker } \varphi_n$ and so to prove that $R(F, K_F)$ is generated by its elements of degree ≤ 3 it will be enough to show that $\text{Ker } \varphi_4$ is generated by the image of

$\text{Ker } \varphi_1 \otimes R_3 + \text{Ker } \varphi_2 \otimes R_2 + \text{Ker } \varphi_3 \otimes R_1 \to \text{Ker } \varphi_4.$

By (5.2), $\text{Ker } \varphi_n = H^0(F-A_1, nK_F-A_1) = H^0(F-A_1, K_{F-A_1} + (n-1)K_F)$.

Since the restriction maps from $H^0(F, nK_F) \to H^0(F-A_1, nK_F)$ are also surjective (by I.6.9), it will be enough to show that (letting $S_n = H^0(F-A_1, nK_F)$) $\operatorname{Ker}\varphi_4$ is generated by the images of $\operatorname{Ker}\varphi_1 \otimes S_3 + \operatorname{Ker}\varphi_2 \otimes S_2 + \operatorname{Ker}\varphi_3 \otimes S_1$.

Case $\ell = 1$. If $D = F - A_1$ is 2-connected (i.e. the standard decomposition of F is $F = A_1 + D$), K_{F-A_1} is generated by its global sections and hence by Castelnuovo's lemma $\operatorname{Ker}\varphi_1 \otimes S_3 \to \operatorname{Ker}\varphi_4$ is surjective if $H^1(D, 3K_F-K_D) = 0$. By duality

$$h^1(D, 3K_F- K_D) = h^0(D, 2K_D-3K_F).$$

This last sheaf has negative degree on D and for each $\Gamma \subset D$, $\deg_\Gamma(2K_D-3K_F) = = (-K- F-2A_1)\Gamma$.

So if D has no common components with A_1 we have the result. Otherwise D and A_1 have common components and by (4.4) $A_1 = E$ is an elliptic tail appearing with multiplicity 1 in D. Now, for every (-2)-curve θ in E, $\theta E = 0$. Also the unique component C_0 of E, such that $\deg_{C_0} K_F = 1$, appears with multiplicity 1 in D and $-(K+F+2E)C_0 = 1$. Then,since $\deg_D(2K_D-3K_F) = -5 < 0$ and D is 2-connected, by(I.3.2) the sheaf $\mathcal{O}_D(2(K+D)-3K_F)$ cannot have global sections .Thus $h^1(D, 3K_F- K_D) = 0$.

Case $\ell \geq 2$. In this case K_{F-A_1} will not be generated by its global sections and we have to make a further decomposition. It is easy to see that the restriction maps $f_n : H^0(F-A_1, nK_F-A_1) \to H^0(E_2, nK_F-A_1)$ are surjective for every n and by (5.2), $\operatorname{Ker} f_n = H^0(F-A_1-A_2, nK_F-A_1-A_2)$. Then we can prove that $\operatorname{Ker}\varphi_4$ is generated by the image of $\operatorname{Ker}\varphi_1 \otimes R_3 + \operatorname{Ker}\varphi_2 \otimes R_2 + \operatorname{Ker}\varphi_3 \otimes R_1$, by showing that

$$H^0(E_2, K_F-A_1) \otimes R_3 + H^0(E_2, 2K_F-A_1) \otimes R_2 \to H^0(E_2, 4K_F-A_1)$$

is surjective and that

$$\operatorname{Ker} f_1 \otimes R_3 + \operatorname{Ker} f_2 \otimes R_2 + \operatorname{Ker} f_3 \otimes R_1 \to \operatorname{Ker} f_4$$

is surjective.

By the properties of a standard decomposition $E_2 A_1 = 0$, hence $\deg_{E_2}(nK_F-A_1) = n$, and for every (-2)-curve θ in E_2 $\deg_\theta(nK_F-A_1) \geq 0$. If C_0 is the unique component of E such that $KC_0 = 1$, $C_0A_1 \leq 1$. Hence nK_F-A_1 is a sheaf of degree n in E_2 such that $\deg_\Gamma(nK_F-A_1) \geq 0$, for all $\Gamma \subseteq E$.

By Riemann-Roch, $h^0(E_2, nK_F-A_1) = n$ and by (I.7.2) nK_F-A_1 is generated by its global sections for $n > 2$, and for $n = 1$, K_F-A_1 has a unique section s vanishing only on a non-singular point P of E_2.

Then $\operatorname{Im}\{H^0(E_2, K_F-A_1) \otimes R_3 \to H^0(E_2, 4K_F-A_1)\}$ has codimension 1 in $H^0(E_2, 4K-A_1)$. Since both $2K_F$ and $2K_{F-A_1}$ are generated by their global sections there exist $\alpha \in H^0(E_2, 2K_F-A_1)$, $\beta \in H^0(E, 2K_F)$ such that $\alpha(P) \neq 0$, $\beta(P) \neq 0$ and $\alpha\beta \notin \operatorname{Im}\{H^0(E_2, K_F-A_1) \otimes R_3 \to H^0(E_2, 4K_F-A_1)\}$. Hence $H^0(E_2, 4K_F-A_1)$ is generated by the image of

$$H^0(E_2, K_F-A_1) \otimes R_3 + H^0(E_2, 2K_F-A_1) \otimes R_2 .$$

Now we need to look at $\operatorname{Ker} f_n$. If $\ell = 2$ i.e. the standard decomposition of F is $F = A_1 + A_2 + D$ then D is a 2-connected genus 1 divisor and $\operatorname{Ker} f_n = H^0(D,(n-1)K_F)$. Since the restriction map from $R_3 \to H^0(D, nK_F)$ is surjective by (5.4) then $\operatorname{Ker} f_1 \otimes R_3 \to \operatorname{Ker} f_4$ is surjective and we have the result.

If $\ell = 3$, i.e. the standard decomposition of F is $F = A_1+A_2+A_3+D$, we will have to make a further decomposition.

Since by (4.11) A_3-E_3 and E_3 have no common components it is easy to see that the restriction maps

$$g_n\colon \mathrm{Ker}\, f_n = H^0(A_3+C, nK_F-A_1-A_2) \to H^0(C, nK_F-A_1-A_2)$$

are surjective and $\mathrm{Ker}\, g_n = H^0(E_3, (n-1)K_F)$.

Since C is isomorphic to $\mathbb{P}^1$ and $CA_1 = CA_2 = 1$, $\mathcal{O}_C(2K_F-A_1-A_2) \cong \mathcal{O}_C$ and by (5.4) the restriction maps $R_n \to H^0(C, nK_F)$ are surjective for $n \geq 2$, $H^0(C, 2K_F-A_1-A_2) \otimes R_2 \to H^0(C, 4K_F-A_1-A_2)$ is surjective.

Since $\mathrm{Ker}\, g_n = H^0(E_3, \mathcal{O}_{E3})$, $\mathrm{Ker}\, g_1 \otimes R_3 \to \mathrm{Ker}\, g_4$ is also surjective and hence $\mathrm{Ker}\, \varphi_1 \otimes R_3 + \mathrm{Ker}\, \varphi_2 \otimes R_2 \longrightarrow \mathrm{Ker}\, \varphi_4$ is surjective.

Section 6. Properties of $R(F, K_F)$ for a double genus 3 fibre.

(6.1) If F is a genus 3 fibre which is not 1-connected then $F = 2C$, where C is a 1-connected divisor with $C^2 = 0$, $KC = 2$, i.e. $p_a(C) = 2$. By (I.8.8) $\mathcal{O}_C(C)$ is a torsion sheaf of order 2. C has the numerical properties of a genus 2 fibre (section 3).

In this section we prove that $R(F, K_F)$ in this case is also generated by its elements of degree ≤ 3 (Theorem 6.8) and set up the grounds for the calculations in (III.5)

(6.2) Lemma If $F = 2C$ is a double fibre of genus 3 then:

(i) $\quad \mathcal{O}_C(nK_F) = \begin{cases} \mathcal{O}_C(nK_C) & \text{for } n \text{ even} \\ \mathcal{O}_C(nK_C+C) & \text{for } n \text{ odd} \end{cases}$

(ii) $\quad h^1(C, nK_F) = 0$, for all $n \geq 1$.

(iii) $\quad h^1(C, K_C + nK_F) = 0$, for all $n \geq 1$.

Proof From $\mathcal{O}_C(C)$ being a torsion sheaf of order 2 and $\mathcal{O}_C(nK_F) \cong \mathcal{O}_C(nK + 2\,nC) \cong \mathcal{O}_C((nK_C) + nC)$ we have (i).

Now (ii) and (iii) are easy to prove (cf. I.6.8-6.9) using duality, $\deg K_{F|_\Gamma} \geq 0$ and $\deg (K_F - K_C)_{|\Gamma} \geq 0$, for all $\Gamma \subset C$, and (I.3.1).

(6.3) Proposition Let $F = 2C$ be a double fibre of genus 3 and $\varphi_n : H^0(F, nK_F) \longrightarrow H^0(C, nK_F)$ the restriction maps. Then

(i) The maps φ_n are surjective, for all $n \geq 1$.

(ii) $\operatorname{Ker} \varphi_n = H^0(C, K_C + (n-1)\,K_F)$ for all $n \geq 1$.

(iii) The maps $h_n : \operatorname{Im} \varphi_2 \otimes \operatorname{Im} \varphi_n \to \operatorname{Im} \varphi_{n+2}$ are surjective for $n \geq 3$.

(iv) If C is 2-connected the maps

$$f_n : \operatorname{Ker} \varphi_1 \otimes H^0(C, nK_F) \longrightarrow \operatorname{Ker} \varphi_{n+1}$$

are surjective for $n \geq 3$, $\operatorname{Im} f_2$ has codimension 1 and $\operatorname{Im} f_1$ has codimension 2.

Proof Since $\mathcal{O}_C(nK_F - C) \cong \mathcal{O}_C((n-1)K_F + K_C)$ we have (i) and (ii) by considering the cohomology sequence of

$$0 \longrightarrow \mathcal{O}_C(nK_F - C) \longrightarrow \mathcal{O}_F(nK_F) \longrightarrow \mathcal{O}_C(nK_F) \longrightarrow 0$$

and using (6.2).

Since $2K_F$ is generated by its global sections, by (I.8.13),and $h^1(C, nK_F) = 0$, for all $n \geq 1$, we can apply Castelnuovo's lemma and obtain (iii).

Now (iv) comes from the free pencil trick and a dimension count,using the facts that K_C is generated by its global sections (because C is 2-connected by assumption) and

$$h^0(C, nK_F - K_C) = \begin{cases} 0 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ 2n-3 & \text{if } n \geq 3 \end{cases} .$$

(6.4) Lemma Let F be as in (6.3) and $D \subset C$ such that $D(C-D) = 1$. Then the restriction maps

$$r_n : H^0(C, nK_F) \longrightarrow H^0(D, nK_F)$$
$$r'_n : H^0(C, (n-1)K_F + K_C) \longrightarrow H^0(D, (n-1)K_F + K_C)$$

are surjective for $n \geq 1$.

Proof Since C is 1-connected, $C-D$ is 1-connected also by (I.2.2). Because $(C-D)^2 = -1$, $\deg K_{F|_{C-D}} > 0$ and so $\deg K_{C|_{C-D}} > 0$ also.

Now the remainder of the proof is the same as the proofs of (I.6.8) and (I.6.9) and is omitted. The only extra twist is the fact that $h^1(C-D, K_F-D) = 0$, since $\mathcal{O}_C(C) \not\cong \mathcal{O}_C$

(6.5) Proposition Let $F = 2C$ be a double fibre of genus 3. Then C is 2-connected or C can be decomposed as either

Type (i) $C = E + E' + A$ where E and E' are two elliptic tails without common components and either $A = 0$, and $EE' = 1$ or E and E' are disjoint and A is a simple chain of (-2)-curves linking E and E'

or

Type (ii) $C = 2E + B + \theta_0$, where B is a (-2)-cycle with $BE = 1$, θ_0 is a (-2)-curve satisfying $\theta_0 B = 0$ and $\theta_0 E = 1$.

Proof See the proof of (3.2), which only uses the numerical properties of F.

For a picture of C see (3.3).

(6.6) Lemma Let $F = 2C$ be a fibre with C of Type (i) or Type (ii) as in (6.4). Write $\Gamma = E+A$ if C is of Type (i) or $\Gamma = E+\theta_0$ is C is of Type (ii).

Let

$$f_n : H^0(C, nK_F) \longrightarrow H^0(C-E,, nK_F)$$
$$f'_n : H^0(C, nK_F) \longrightarrow H^0(C-\Gamma, nK_F)$$

and

$$g_n : H^0(C, (n-1)K_F + K_C) \longrightarrow H^0(C-E, (n-1)K_F+K_C)$$
$$g'_n : H^0(C, (n-1)K_F+K_C) \longrightarrow H^0(C-\Gamma, (n-1)K_F+K_C)$$

be the restriction maps. Then

(i) $\operatorname{Ker} f_n = \operatorname{Ker} f'_n$ and $\operatorname{Ker} g_n = \operatorname{Ker} g'_n$, for every $n \geq 1$.

(ii) $\operatorname{Ker} g'_1$ is generated by a single element and the maps

$$\operatorname{Ker} g'_1 \otimes H^0(C, nK_F) \longrightarrow \operatorname{Ker} g_{n+1}$$

are surjective for every $n \geq 1$.

(iii) $\dim \operatorname{Ker} f'_n = n-1$, for every $n \geq 1$ and the images of the maps

$$\delta_n : \operatorname{Ker} f'_2 \otimes H^0(C, nK_F) \longrightarrow \operatorname{Ker} f'_{n+2} \text{ have codimension } 1 .$$

Proof (i) $\operatorname{Ker} f_n = H^0(E, nK_F-(C-E))$ and $\operatorname{Ker} f'_n = H^0(\Gamma, nK_F-(C-\Gamma))$. Suppose that $\Gamma-E \neq 0$ (otherwise there is nothing to prove); we have the exact sequence

$$0 \to H^0\big(E, nK_F-(C-E)\big) \xrightarrow{\alpha_n} H^0\big(\Gamma, nK_F-(C-\Gamma)\big) \longrightarrow H^0\big(\Gamma-E, nK_F-(C-\Gamma)\big).$$

Now $\Gamma-E$ is 1-connected and $\deg_{\Gamma-E}\big(nK_F-(C-\Gamma)\big) = -1$. Also by our choice of Γ, $\deg_\theta\big(nK_F-(C-\Gamma)\big) \leq 0$ for each (-2)-curve in θ in $\Gamma=E$, so by (I.3.1) $H^0\big(\Gamma-E, nK_F-(C-\Gamma)\big) = 0$ and therefore α_n is an isomorphism.

The proof of $\mathrm{Ker}\, g_n = \mathrm{Ker}\, g'_n$ is similar and is omitted.

(ii) By (i) $\mathrm{Ker}\, g_1 = \mathrm{Ker}\, g'_1$. Since $\mathrm{Ker}\, g_1 = H^0\big(E, K_C-(C-E)\big) = H^0(E, \mathcal{O}_E)$ it follows that $\mathrm{Ker}\, g_1$ is generated by a single element s. Also by (i) $\mathrm{Ker}\, g_n = \mathrm{Ker}\, g'_n$, for all $n \geq 1$ and so it will be enough to prove (ii) for $\mathrm{Ker}\, g_n$.

The map $\mathrm{Ker}\, g_1 \otimes H^0(C, nK_F) \longrightarrow \mathrm{Ker}\, g_{n+1} = H^0(E, nK_F)$ factors through $\mathrm{Ker}\, g_1 \otimes H^0(C, nK_F) \longrightarrow \mathrm{Ker}\, g_1 \otimes H^0(E, nK_F) \longrightarrow \mathrm{Ker}\, g_{n+1}$ and thus (ii) is clear since $\mathrm{Ker}\, g_1 = H^0(E, \mathcal{O}_E)$ and the restriction maps

$$H^0(C, nK_F) \longrightarrow H^0(E, nK_F)$$

are surjective by (6.4).

(iii) Again it will be enough to prove the statement for $\mathrm{Ker}\, f_n$. Now

$$\mathrm{Ker}\, f_n = H^0\big(E, nK_F-(C-E)\big) = H^0\big(E, \mathcal{O}_E\big((n-1)K_F+C\big)\big).$$

Since $\deg K_{F|\Gamma} \geq 0$, for every component Γ of C and $\mathcal{O}_E(C) \not\cong \mathcal{O}_E$ we have by Riemman Roch that $\dim \mathrm{Ker}\, f'_n = n-1$. The second part of the statement is trivial because $\mathrm{Ker}\, f_2$ is generated by a single element that vanishes only in a nonsingular point P of E. Using (6.4) and the fact that $\mathrm{Ker}\, f_2 \otimes H^0(C, nK_F) \longrightarrow \mathrm{Ker}\, f_{n+2}$ factors through $\mathrm{Ker}\, f_2 \otimes H^0(C, nK_F) \longrightarrow \mathrm{Ker}\, f_2 \otimes H^0(E, nK_F) \longrightarrow \mathrm{Ker}\, f_{n+2}$ we have (iii).

(6.7) Lemma If C is of type (ii) as in (6.5) and $\Gamma = E + \theta_0$ then the restriction maps

$$\alpha_n : H^0(B+E, nK_F) \longrightarrow H^0(E, nK_F)$$

and $\quad \beta_n : H^0(B+E, (n-1)K_F+K_C) \longrightarrow H^0(E, (n-1)K_F+K_C)$

are isomorphisms for $n \geq 1$.

Proof We have $\mathrm{Ker}\, \alpha_n = H^0(B, nK_F-E)$. Since $\deg_B (nK_F-E) = -1$, B is

1-connected and $\theta(nK_F) = 0$, $\theta E \geq 0$ for every (-2)-curve θ in B, we can apply lemma I.3.1 and obtain $\mathrm{Ker}\,\alpha_n = 0$. Hence α_n is an isomorphism by a dimension count.

The statement for β_n can be proved in a similar way and we omit the proof.

(6.8) Lemma Let f_n, g_n be as in (6.6). Then the maps

(i) $\Delta_n : \mathrm{Ker}\, f_2 \otimes H^0(C, nK_F) + \mathrm{Ker}\, f_3 \otimes H^0(C, (n-1)K_F) \to \mathrm{Ker}\, f_{n+2}$

(ii) $\Gamma_n : \mathrm{Im}\, g_1 \otimes H^0(C, nK_F) + \mathrm{Im}\, g_2 \otimes H^0(C, nK_F) \to \mathrm{Im}\, g_{n+1}$

are surjective for $n \geq 2$.

Proof Let E, E' be as in (6.5) (with $E = E'$ if C is of Type (ii)). Then by (6.6), (6.7) we have $\mathrm{Ker}\, f_1 = 0$, $\mathrm{Ker}\, f_n = H^0(E, (n-2)K_F + K_C)$, for $n \geq 2$ and $\mathrm{Im}\, g_n = H^0(E', (n-1)K_F + K_C)$, for $n \geq 1$.

By (6.4) the restriction maps

$$H^0(C, nK_F) \longrightarrow H^0(E, nK_F)$$
$$H^0(C, nK_F) \longrightarrow H^0(E', nK_F)$$

are surjective, and thus it will be enough to prove (i) and (ii) with C substituted by E and E' respectively.

(i) We have $\dim \mathrm{Ker}\, f_2 = 1$ and by (I.7.2) a generator of $\mathrm{Ker}\, f_2$ vanishes only at a nonsingular point $P \in E$. By (6.6.(ii)) $\mathrm{Im}\{\mathrm{Ker}\, f_2 \otimes H^0(E, nK_F) \to \mathrm{Ker}\, f_{n+2}\}$ has codimension 1 in $\mathrm{Ker}\, f_{n+2}$ and thus to prove (i) it is enough to show that there exists $u \in \mathrm{Ker}\, f_3$, $v \in H^0(E, (n-1)K_F)$ such that $u(P) \neq 0$, $v(P) \neq 0$

The existence of such an u comes from $\deg_E(K_F + K_C) = 2$ and (I.7.2), and the existence of such a v comes from $\mathcal{O}_E(K_F) \not\cong \mathcal{O}_E(K_C)$ giving that

$\varepsilon \in H^0(E, K_F)$, $\varepsilon \neq 0$ implies $\varepsilon(P) \neq 0$. Taking $v = \varepsilon^{n-1}$ we have $v(P) \neq 0$, and so (i).

(ii) We have $\dim \operatorname{Im} g_n = n$ and again by (I.7.2) $\operatorname{Im} g_1$ is generated by one element vanishing only in P such that $\mathcal{O}_E(P) \cong \mathcal{O}_E(K_C)$. Since

$$\operatorname{Im}\{\operatorname{Im} g_1 \otimes H^0(E, nK_F) \longrightarrow \operatorname{Im} g_{n+1}\}$$

has codimension 1 in $\operatorname{Im} g_{n+1}$, we use the same argument as in (i).

(6.9) **Theorem** If $F = 2C$ is a double fibre of genus 3 , $R(F, K_F)$ is generated by its elements of degree ≤ 3 .

Proof Let φ_n be the restriction maps $H^0(F, nK_F) \longrightarrow H^0(C, nK_F)$ and consider the exact sequences

$$0 \to H^0(C, (n-1)K_F+K_C) \to H^0(F, nK_F) \to H^0(C, nK_F) \to 0 .$$

By (I.9.2), since the maps φ_n are surjective for every n , it will be enough to show that

(A) $\operatorname{Ker} \varphi_1 \otimes H^0(C, nK_F) + \operatorname{Ker} \varphi_2 \otimes H^0(C, (n-1)K_F \to \operatorname{Ker} \varphi_{n+1}$ is surjective for $n \geq 3$ and by (6.3.iii) that

(B) $\operatorname{Im} \varphi_1 \otimes H^0(C, 3K_F) + \operatorname{Im} \varphi_2 \otimes H^0(C, 2K_F) \longrightarrow \operatorname{Im} \varphi_4$ is surjective.

We will consider separately the case when C is 2-connected and C is 1-connected.

Case 1. C is of type (i) or (ii) as in (6.5) In this case we can decompose $\operatorname{Ker} \varphi_n$ as

$$0 \to \operatorname{Ker} g_n = H^0(E, (n-1)K_F) \longrightarrow \operatorname{Ker} \varphi_n \longrightarrow \operatorname{Im} g_n = H^0(E', (n-1)K_F+K_C) \to 0,$$

and we can decompose $\operatorname{Im} \varphi_n$ for $n \geq 2$ as

$$0 \to \operatorname{Ker} f_n = H^0(E, (n-2)K_F+K_C) \longrightarrow \operatorname{Im} \varphi_n \longrightarrow H^0(E', nK_F) \to 0 .$$

Using (I.9.2) , (A) follows from (6.6.ii) and (6.8.ii) and (B) from (6.8.i) and the description of $R(E', K_F)$ (K_F is a sheaf of degree 1 on E') in (I.7.2).

Case 2. C is 2-connected.

In this case (A) is proved in (6.3.iv). For (B) we will have to analyse how $H^0(C, K_F)$ and $H^0(C, 2K_F)$ are generated. Now because C is 2-connected it is clear that if ε is a generator of $H^0(C, K_F)$, $\varepsilon_{|\Gamma} \neq 0$, for every $\Gamma \subset C$ and thus ε vanishes only at isolated points of each component. Thus $\varepsilon^2 \neq 0$. Suppose that (s_0, s_1) form a basis of $H^0(C, K_C)$. Since $\mathcal{O}_C(2K_F) \cong \mathcal{O}_C(2K_C)$ and $(s_0{}^2, s_0 s_1, s_1{}^2)$ form a basis of $H^0(C, 2K_C)$ by (I.9.4), necessarily $\varepsilon^2 = \alpha_0\, s_0{}^2 + \alpha_1\, s_0 s_1 + \alpha_2\, s_1{}^2$. Since $\mathcal{O}_C(K_F) \not\cong \mathcal{O}_C(K_C)$, we cannot have simultaneously $\alpha_0 = \alpha_1 = 0$ or $\alpha_1 = \alpha_2 = 0$. Thus, by choosing s_0, s_1 conveniently, we can assume that $\varepsilon^2 = \lambda\, s_0 s_1$, with $\lambda \neq 0$. We have then two new generators for $H^0(C, 2K_F)$ corresponding to $s_0{}^2, s_1{}^2$.

Now $H^0(C, 3K_F)$ is 5-dimensional and thus $V = \varepsilon\, H^0(C, 3K_F) \hookrightarrow H^0(C, 4K_F)$ is 5-dimensional. Since s_0, s_1 have no common zeros neither $s_1{}^4$, $s_0{}^4 \in V$. Otherwise we would have linear relations of the form

$$\varepsilon\gamma = \delta_0 s_1{}^4 + \delta_1\, s_0{}^4 \text{ with } \delta_0 \neq 0 \text{ or } \delta_1 \neq 0$$

and this is impossible because ε vanishes on the zeros of s_0 and s_1 and s_0, s_1 have no common zeros. Thus $V + \langle s_0{}^4, s_1{}^4 \rangle = H^0(C, 4K_F)$ and we have (B).

CHAPTER III.

Calculation of $R(F, K_F)$ for a genus 3 fibre.

Section 1. Analytic decomposition of F .

(1.0) In this chapter we are going to describe explicitly the k-algebra $R(F,K_F)$ of a genus 3 fibre F , by generators and relations. We start by dealing with the case when F is 1-connected but not 2-connected, i.e. a standard numerical decomposition of F , $F = \sum_{i=1}^{\ell} A_i + D$ as in (II 4.2) has $\ell \geq 1$.

In this section we are going to show that each of these fibres falls in one of three distinct cases (Def.1.5,1.13,1.15), and find the number and degrees of a minimal set of generators for $R(F, K_F)$ in each case (see Theorem 1.18). In sections 2, 3, and 4 we describe explicitly $R(F, K_F)$ for each of these cases.

In this section we also establish that we can use the technique explained in (II.1) to describe $R(F, K_F)$.

(1.1) **Notation** In this section F stands for 1-connected genus 3 fibre which is not 2-connected. Given an elliptic cycle A_i , E_i stands for the elliptic tail contained in A_i .

(1.2) Proposition Let $F = \sum_{i=1}^{\ell} A_i + D$ be a standard numerical decomposition of F and let E_1 be the elliptic tail contained in A_1. Let $\varphi : R(F, K_F) \to S = R(F-A_1, K_F)$ be the morphism induced by the restriction maps. Then

(i) φ is surjective.

(ii) $\operatorname{Ker}\varphi$ is a principal ideal generated by an element s of degree 1 such that $s_{|F-E_1} \equiv 0$.

(iii) The ideal $J = \{x \in R : sx = 0\}$ is generated by its homogeneous elements of degree < 6.

Proof (i) is just proposition (II.5.4)

(ii) $\operatorname{Ker}\varphi = \oplus \operatorname{Ker}\varphi_n$, where $\varphi_n : R_n \to H^0(F-A_1, nK_F)$ are the restriction maps. By (II.5.2) $\operatorname{Ker}\varphi_n = H^0(A_1, nK_F-(F-A_1)) = H^0(E_1, nK_F-(F-E_1)) = H^0(E_1, (n-1)K_F)$. In particular $\operatorname{Ker}\varphi_1 = H^0(E_1, \mathcal{O}_{E_1})$.

Since the restriction maps $R_n \to H^0(E_1, nK_F)$ are surjective and $\operatorname{Ker}\varphi_1 \otimes R_n \to \operatorname{Ker}\varphi_{n+1}$ factors as

$$\operatorname{Ker}\varphi_1 \otimes R_n \to \operatorname{Ker}\varphi_1 \otimes H^0(E_1, nK_F) \to \operatorname{Ker}\varphi_{n+1}$$

we have $\operatorname{Ker}\varphi_n = \operatorname{Ker}\varphi_1 . R_n$, for every $n \geq 1$. This proves the assertion.

(iii) The ideal J is the kernel of the linear map of graded algebras $R \to R$ given by $y \to sy$ and thus it is a homogeneous ideal. Also $J \cap R_n = \operatorname{Ker}\{H^0(F, nK_F) \to H^0(E_1, nK_F)\} = H^0(F-A_1, nK_F-A_1)$.

Since $H^0(F, 2K_F) \to H^0(F-A_1, 2K_F)$ is onto, and we know that $2K_F$ is free and the map $H^0(F-A_1, nK_F-A_1) \otimes H^0(F,2K_F) \to H^0(F-A_1, (n+2)K_F-A_1)$

factors through $H^0(F-A_1, nK_F-A_1) \otimes H^0(F-A_1, 2K_F)$, using Castelnuovo's lemma we obtain $J_n.R_2 = J_{n+2}$ for $n \geq 4$ .This proves (iii).

(1.3) **Remark** Using this proposition and the technique described in (II.1), it will be easy to recover R once S is known. Let $S_n = H^0(F-A_1, nK_F)$ (so that $S = \oplus S_n$) . We have $\dim S_n = 3n-1$.

(1.4) **Proposition** Let γ be the map from $S_1 \otimes S_1 \to S_2$. Then

$$\operatorname{rk}\gamma = \begin{cases} 2 & \text{if } F-A_1 \text{ contains an elliptic tail E such that } \mathcal{O}_E(A_1) \cong \mathcal{O}_E \\ 3 & \text{otherwise} \end{cases} .$$

Proof Suppose that s_1, s_2 are two non-zero elements of S_1 such that $s_1 s_2 = 0$ and let Z_{s_i} be the biggest divisor contained in $F-A_1$ where $s_i \equiv 0$. Then $Z_{s_1} + Z_{s_2} \supset F-A_1$. Let $D_{s_i} = F-A_1-Z_{s_i}$ for $i = 1,2$. By (I.3.1) we have

$$1 \leq Z_{s_i} . D_{s_i} \leq \deg K_{F|D_{s_i}} .$$

Since $\deg K_{F|F-A_1} = 3$, we can assume without loss of generality that $\deg K_{F|Z_{s_1}} = 2$. Then $\deg K_{F|Z_{s_2}} \geq 1$, $\deg K_{F|D_{s_1}} = 1$ and $D_{s_1} Z_{s_1} = 1$. Since $(F-A_1)^2 = -1$, D_{s_1} is an elliptic cycle, and $s_1 \in \operatorname{Ker}\{S_1 \to H^0(Z_{s_1}, K_F)\} = H^0(D_{s_1}, K+D_{s_1}+A_1)$. So D_{s_1} contains an elliptic tail E such that $\mathcal{O}_E(A_1) \cong \mathcal{O}_E$ (by II.5.2, II.5.3).

Let s be a generic element of S_1 (generic in the sense that if C is a component of $F-A_1$ such that $s_{|C} \equiv 0$, then C is a fixed component of K_F). Then $sx \neq 0$ for any x in S_1 . Otherwise, since $Z_s \subset Z_x$ and $Z_x + Z_s \supset D_x$,

D_x would be an elliptic cycle contained in the fixed locus of K_F. This is impossible and thus if x,y are independent elements of S_1, sx, sy are independent elements of S_2. So $\mathrm{rk}\,\gamma \geq 2$.

Let $\langle x,y \rangle$ be a basis of S_1. If $\mathrm{rk}\,\gamma = 2$ then x^2, xy, y^2 are not independent in S_2 and thus there exist $s_1, s_2 \in S_1$ such that $s_1 s_2 = 0$. By the considerations above, $F-A_1$ contains an elliptic tail E such that $\mathcal{O}_E(A_1) \cong \mathcal{O}_E$.

Suppose now that $F-A_1$ contains an elliptic tail E with $\mathcal{O}_E(A_1) \cong \mathcal{O}_E$. If $h_n : H^0(F-A_1, nK_F) \to H^0(F-A_1-E, nK_F)$ are the restriction maps, then $\mathrm{Ker}\,h_1 = H^0(E, K+E+A_1)$ and $\mathrm{Ker}\,h_2 = H^0(E, K_F)$ are both 1-dimensional. So, if $s \in \mathrm{Ker}\,h_1$ and s,t generate S_1, both s^2, $st \in \mathrm{Ker}\,h_2$, and thus $\mathrm{rk}\,\gamma = 2$.

(1.5) **Definition** F is *analytically of type* I if there is a standard decomposition $F = \sum_{i=1}^{k} A_i + D$ such that $F-A_1$ does not contain any elliptic tail E with $\mathcal{O}_E(A_1) \cong \mathcal{O}_E$.

(1.6) **Lemma** F is analytically of type I if and only if for any standard numerical decomposition $F = \sum_{i=1}^{k} B_i + D$, $(F-B_1)$ does not contain an elliptic tail E with $\mathcal{O}_E(B_1) \cong \mathcal{O}_E$.

Proof This is clear by looking at the propositions on standard decompositions and bearing in mind that if A_i and A_j are disjoint then $\mathcal{O}_{Ej}(A_j) \cong \mathcal{O}_{Ej}$.

(1.7) Proposition F is analytically of type I if and only if for any standard numerical decomposition $F = \sum_{i=1}^{n} A_i + D$ either

(i) $n = 1$.

or

(ii) $n = 2$, $A_1 \supset A_2$ and $\mathcal{O}_E(A_1) \not\cong \mathcal{O}_E$ if E is the common elliptic tail in A_1 and A_2

or

(iii) $n = 3$, $A_1 \supset A_2 \supset A_3$ and $\mathcal{O}_E(A_1) \not\cong \mathcal{O}_E$ if E is the elliptic tail in A_1, A_2, A_3.

Proof This is again clear by the properties of standard numerical decompositions and (II.5.3).

(1.8) Remark If $F-A_1$ contains an elliptic tail E such that $\mathcal{O}_E(A_1) \cong \mathcal{O}_E$, then $E(F-A_1-E) = 1$ and so F admits a standard numerical decomposition $F = \sum_{i=1}^{\ell} A_i+D$, with $\ell \geq 2$ .The converse is false.

(1.9) Proposition Let $F = \sum_{i=1}^{\ell} A_1+D$ be a standard numerical decomposition of F with $\ell \geq 2$, E the elliptic tail contained in A_2 and suppose that $\mathcal{O}_E(A_1) \cong \mathcal{O}_E$. Let $\varphi_n : H^0(F-A_1, nK_F) \to H^0(F-A_1-A_2, nK_F)$ be the restriction maps and $\varphi : S = R(F-A_1, K_F) \to R(F-A_1-A_2, nK_F)$ be the morphism induced by the maps φ_n. Then

(i) Im φ_1 has codimension 1 in $H^0(F-A_1-A_2, nK_F)$ and φ_n is surjective for $n \geq 2$.

(ii) Ker φ is a principal ideal of S generated by an element s of S_1 such that $s_{|F-A_1-E_2} \equiv 0$.

(iii) The ideal $J = \{x \in S : sx = 0\}$ is generated by its homogeneous elements of degree < 6.

Proof (i) is just proposition (II.5.4).

The proofs of (ii) and (iii) are similar to those of (ii) and (iii) of (1.2) and we omit them.

(1.10) **Proposition** Let φ, F be as in (1.9), $T = \mathrm{Im}\,\varphi \subset R(F-A_1-A_2,K_F)$, $T_n = \mathrm{Im}\,\varphi_n$ and f the map $T_1 \otimes T_2 \to T_3$. Then

$$\mathrm{rk}\, f = \begin{cases} 2 & \text{if } F-A_1-A_2 \text{ contain an elliptic tail E such that } \mathcal{O}_E(A_1+A_2) \cong \mathcal{O}_E \\ 4 & \text{otherwise} \end{cases}$$

Proof Let y be a generator of T_1 and $\langle x_0, x_1, x_2, x_3 \rangle$ a basis for T_2. Then either the elements yx_i are independent in T_3 (in which case Im f has dimension 4) or there exists $t \in T_2$ such that $yt = 0$.

For $x \in T_1$ or T_2 let Z_x be the largest divisor contained in $F-A_1-A_2$ where $x \equiv 0$, and let $D_x = F-A_1-A_2-Z_x$. If $yt = 0$ then $Z_y + Z_t \supset F-A_1-A_2$ and thus by lemma (I.3.1)

$$Z_t D_t \leq \deg K_{F|D_t}$$

$$Z_y D_y \leq \deg K_{F|D_y} .$$

Since $F-A_1-A_2$ is 1-connected and $\deg K_{F|_{F-A_1-A_2}} = 2$, we have, e.g., $Z_y D_y = 1$ and $\deg K_{F|_{D_y}} = 1$. Then either Z_y or D_y is an elliptic cycle. Since an elliptic cycle can not be contained in the fixed locus of K_F, D_y is an elliptic cycle. If E is the elliptic tail contained in D_y, using a similar argument to that in (1.3) we get $\mathcal{O}_E(A_1+A_2) \cong \mathcal{O}_E$, and $\operatorname{Im} f$ is 2-dimensional.

On the other hand if $F-A_1-A_2$ contains an elliptic tail E such that $\mathcal{O}_E(A_1+A_2) \cong \mathcal{O}_E$, also with a similar argument to the one in (1.3), we obtain that $\operatorname{Im} f$ has dimension 2.

(1.11) Remark If $\operatorname{rk} f = 2$ (f as in 1.10) then $F-A_1-A_2$ contains an elliptic tail E such that $\mathcal{O}_E(A_1+A_2) \cong \mathcal{O}_E$ and $E(F-A_1-A_2-E) = 1$. Thus a standard numerical decomposition of F has $n = 3$ and $F-A_1-A_2$ can be written as $A_3 + C$ with $C \cong \mathbb{P}^1$.

(1.12) Proposition Let F, T, f be as in (1.10) and assume that $F-A_1-A_2 = A_3 + C$ with $\mathcal{O}_{E_3}(A_1+A_2) \cong \mathcal{O}_{E_3}$ (i.e. $\operatorname{rk} f = 2$ by (1.10)).

Let $\psi : T \to R(C, K_F) = R(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ be the morphism induced by the restriction maps. Then

(i) $\operatorname{Ker} \psi = T_1.T$ and $\operatorname{Ker} \psi$ is generated by $y \in T_1$ such that $y_{|A_3+C-E_3} \equiv 0$.

(ii) $\operatorname{Im} \psi$ is the subring of $R(\mathbb{P}^1, \mathcal{O}(1))$ generated by its elements of degree ≥ 2.

(iii) $J = \{s \in T : sy = 0\}$ is generated by its homogeneous elements of degree < 6.

Proof The proof is similar to the proofs of (1.10), (1.8) and we omit it. We just remark that by (II.5.4) the restriction map $H^0(F, nK_F) \to H^0(C, nK_F)$ is the zero map for $n = 1$ and is surjective otherwise.

(1.13) Definition F is *analytically of type* II if there is a standard numerical decomposition $F = \sum_{i=1}^{k} A_i + D$ with $k \geq 2$, such that $\mathcal{O}_{E_2}(A_1) \cong \mathcal{O}_{E_2}$ and $F-A_1-A_2$ does not contain any elliptic tail E' with $\mathcal{O}_{E'}(A_1+A_2) \cong \mathcal{O}_{E'}$.

(1.14) Lemma F is analytically of type II if and only if any standard decomposition $F = \sum_{i=1}^{n} A_i + D$ has $n \geq 2$ and either

(i) $n = 2$, A_1 and A_2 are disjoint

or

(ii) $n = 2$, $A_1 \supset A_2$ and $\mathcal{O}_{E_2}(A_1) \cong \mathcal{O}_{E_2}$

or

(iii) $n = 3$, $A_1 \supset A_i$, A_1 and A_j are disjoint for $\{i,j\} = \{2,3\}$ and $\mathcal{O}_{E_i}(A_1) \not\cong \mathcal{O}_{E_i}$.

Proof This is clear by the uniqueness of the A_i appearing in a standard decomposition, and (II-4.8, 4.15, 5.4).

(1.15) Definition F is *analytically of type* III if there is a standard decomposition of F , $F = \sum_{i=1}^{3} A_i + D$ such that $\mathcal{O}_{E_2}(A_1) \cong \mathcal{O}_{E_2}$ and $\mathcal{O}_{E_3}(A_1+A_2) \cong \mathcal{O}_{E_3}$.

(1.16) Lemma F is analytically of type III if and only if any standard numerical decomposition

$$F = \sum_{i=1}^{3} A_i + D$$

is such that

(i) $A_i \cap A_j = \emptyset$

or

(ii) $A_1 \supset A_2 \supset A_3$ and $\mathcal{O}_{E_3}(A_1+A_2) \cong \mathcal{O}_{E_3}(A_1) \cong \mathcal{O}_{E_3}$;

or

(iii) $A_i \supset A_j$ with $\mathcal{O}_{E_j}(A_i) \cong \mathcal{O}_{E_j}$ and $A_i \cap A_k = \emptyset$,where (i,j,k) is some permutation of (1,2,3) .

Proof This is clear by the properties of standard decompositions.

(1.17) Proposition If F is analytically of type I , Im $\{R_1 \otimes R_2 \longrightarrow R_3\}$ has codimension 1 in R_3 .

Proof Let E be the elliptic tail contained in A_1 . By considering the restriction maps $\varphi_n : R_n \longrightarrow H^0(E, nK_F)$, and recalling the description of $R(E, K_F)$ in (I.7.3), it is clear that it is enough to show that the map

$$\text{Ker}\,\varphi_1 \otimes R_2 + \text{Ker}\,\varphi_2 \otimes R_1 \longrightarrow \text{Ker}\,\varphi_3$$

is surjective.

Now $\text{Ker}\,\varphi_n = H^0(F-A_1, nK_F-A_1)$ and by (II.5.4) the maps

$$R_n \to H^0(F-A_1, nK_F)$$

are surjective. Thus it will be enough to show that

$$\text{Ker}\,\varphi_1 \otimes H^0(F-A_1, 2K_F) + \text{Ker}\,\varphi_2 \otimes H^0(F-A_1, K_F) \longrightarrow \text{Ker}\,\varphi_3$$

is surjective.

We consider separately the different cases for a standard decomposition of F, $F = \sum_{i=1}^{n} A_i + D$.

Case 1 $\quad$ **n = 1**

In this case, by (I.4.4) either the standard decomposition of F, $F = A_1+D$ is such that A_1 and D have no common components or $F = E+D$ and $E \subset D$ is an elliptic tail. We consider the two cases separately.

Case 1(i) $\quad F = A_1+D$, $A_1 \cap D = \{P\}$.

Then D is 2-connected, P is a non-singular point of D and $\mathcal{O}_D(K_F) \cong \mathcal{O}_D(K_D+P)$. Since K_D is generated by its global sections and $H^1(D, 2K_F - K_D) = 0$, the map

$$\text{Ker}\,\varphi_1 \otimes H^0(D, 2K_F) \longrightarrow \text{Ker}\,\varphi_3$$

is surjective by Castelnuovo's lemma.

Case 1(ii) $\quad F = E+D$, $E \subset D$

In this case $\mathcal{O}_D(K_D)$ is again generated by its global sections, but because $E \subset D$

and $E(D-E) = 2$, we have $h^1(D, 2K_F-K_D) = 1$. Thus we can not apply Castelnuovo's lemma. Nevertheless, by Riemann-Roch $h^0(D, 2K_F-K_D) = 4$ and using the free pencil trick, we see that

$$\mathrm{Im}\,\{\mathrm{Ker}\,\varphi_1 \otimes H^0(D, 2K_F) \longrightarrow \mathrm{Ker}\,\varphi_3\}$$

has codimension 1 in $\mathrm{Ker}\,\varphi_3$. We will show that there exist $s \in \mathrm{Ker}\,\varphi_2$, $t \in R_1$ such that st is not in this image.

It is easy to verify that the restriction maps

$$g_n : \mathrm{Ker}\,\varphi_n = H^0(D, K_D+(n-1)K_F) \longrightarrow H^0(E, K_D+(n-1)K_F)$$

are surjective. Since K_D is nef and $\deg \mathcal{O}_E(K_D) = 2$, $\mathcal{O}_E(K_D+(n-1)K_F)$ is generated by its global sections and $h^0(E, K_D+(n-1)K_F) = 2+(n-1)$.

Let x_0, x_1 be a basis of $\mathrm{Ker}\,\varphi_1$, and $x_2 \in R_1$ such that (x_0, x_1, x_2) is a basis of R_1. Then x_0, x_1 generate $\mathrm{Im}\,g_1$ and $x_{2|E}$ generates $H^0(E, K_F)$. $\mathrm{Im}\,g_2$ is 3-dimensional and will be generated by x_0x_2, x_1x_2 and a new element β such that $\beta(P) \neq 0$, where P is the unique zero of x_2. Using the free pencil trick and the fact that $\mathcal{O}_E(2K_F-K_D) \cong \mathcal{O}_E(K_F+E) \cong \mathcal{O}_E$, we have that $\mathrm{Im}\,\{x_0.H^0(E, 2K_F) + x_1.H^0(E, 2K_F) \longrightarrow H^0(E, 2K_F+K_D)\}$ has codimension 1.

Because x_0, x_1 have no common zeros and $\mathcal{O}_E(2K_F)$ is generated by its global sections it is easy to verify that $x_2\beta$ is not in that image and hence the statement is proved, for this case.

Case 2 $\;n \geq 2$

In this case F contains a unique elliptic tail E and $A_i \subset A_j$, $i \geq j$. The restriction maps

$$f_n : \mathrm{Ker}\,\varphi_n = H^0(F-A_1, K_{F-A_1}+(n-1)K_F) \longrightarrow H^0(E, K_{F-A_1}+(n-1)K_F)$$

are such that $\mathrm{Ker}\,f_n = H^0(F-A_1-A_2, K_{F-A_1-A_2}+(n-1)K_F)$.

Since K_F is nef, $h^1(F-A_1-A_2, K_{F-A_1-A_2}+(n-1)K_F) = 0$ for $n \geq 2$, and thus the maps f_n are surjective, for all $n \geq 1$.

Also, since K_{F-A_1} is nef and K_F is nef, for $n \geq 2$, $h^0(\mathcal{O}_E(K_{F-A_1}+(n-1)K_F)) = n$ .Also $\mathcal{O}_E(K_{F-A_1}+(n-1)K_F)$ is generated by its global sections and for $n = 1$, $\mathcal{O}_E(K_{F-A_1}+(n-1)K_F) = \mathcal{O}_E(Q)$, with Q a non-singular point of E.

Since $\mathcal{O}_E(A_1) \not\cong \mathcal{O}_E$ necessarily $\mathcal{O}_E(K_{F-A_1}) \not\cong \mathcal{O}_E(K_F)$ and thus if P is the point such that $\mathcal{O}_E(K_F) = \mathcal{O}_E(P)$, we have $P \neq Q$.

To prove the statement it will then be enough to show that

(A) $\quad \operatorname{Ker} f_3 \subset \operatorname{Im} \{\operatorname{Ker} \varphi_1 \otimes R_2 + \operatorname{Ker} \varphi_2 \otimes R_1 \longrightarrow \operatorname{Ker} \varphi_3\}$

and

(B) $\quad \operatorname{Im} f_1 \otimes H^0(E, 2K_F) + \operatorname{Im} f_2 \otimes H^0(E, K_F) \longrightarrow \operatorname{Im} f_3$ is surjective

Now (B) is obvious since $\operatorname{Im} f_1 = H^0(\mathcal{O}_E(Q))$, $\operatorname{Im} f_2 = H^0(\mathcal{O}_E(P+Q))$, $\mathcal{O}_E(K_F) = \mathcal{O}_E(P)$, both $\mathcal{O}_E(2P)$, $\mathcal{O}_E(P+Q)$ are generated by its global sections and $\mathcal{O}_E(2P) \not\cong \mathcal{O}_E(P+Q)$.

For (A) we will have to analyse the possible cases.

If the standard numerical decomposition of F has $n = 2$, i.e. $F = A_1 + A_2 + D$, then, since D is a 2-connected genus 1 divisor, $K_D = \mathcal{O}_D$. Thus $\operatorname{Ker} f_1 = H^0(D, \mathcal{O}_D)$ and so (because the map $R_2 \to H^0(D, 2K_F)$ is surjective by (II.5.4)), the map

$$\operatorname{Ker} f_1 \otimes R_2 \longrightarrow \operatorname{Ker} f_3$$

is surjective. Thus (A) holds and the statement is proved for this case.

If the standard decomposition of F has $n = 3$, we make a further restriction considering the maps

$g_n : \operatorname{Ker} f_n = H^0(A_3+C, K_{A_3+C}+(n-1)K_F) \longrightarrow H^0(E, K_{A_3+C}+(n-1)K_F)$

which are surjective because, by (II.5.2), $\operatorname{Ker} g_n = H^0(C, K_C+(n-1)K_F)$, and $\deg K_{F|C} = 1$. Now $\operatorname{Ker} g_1 = 0 = \operatorname{Ker} g_2$ and $\operatorname{Ker} g_3$ is 1-dimensional.

The restriction map $R_1 \to R(A_3+C, K_F)$ is surjective by (II.5.3) and since $\operatorname{Ker} \{H^0(A_3+C, K_F) \to H^0(E, K_F)\} = H^0(C, K_F-A_3) = H^0(C, \mathcal{O}_C)$ there exists a section $s \in R_1$ such that $s_{|E} = 0$ and $s_{|C} \neq 0$. Since $C \cong \mathbb{P}^1$, we have $s^3 \neq 0$ and thus s^3 generates $\operatorname{Ker} g_3$.

If we now prove that we can find a complementary basis of s^3 in $\operatorname{Ker} f_3$ contained in $\operatorname{Im} \{\operatorname{Ker} \varphi_1 \otimes R_2 + \operatorname{Ker} \varphi_2 \otimes R_1 \longrightarrow \operatorname{Ker} \varphi_3\}$ we have proved (A) and thus the statement. So it will be enough to find three independent elements of $\operatorname{Ker} f_3 \cap \operatorname{Im} \{\operatorname{Ker} \varphi_1 . R_2 + \operatorname{Ker} \varphi_2 . R_1 \longrightarrow \operatorname{Ker} \varphi_3\}$ that restricted to E generate $\operatorname{Im} g_3$.

If $\mathcal{O}_E(A_1+A_2) \not\cong \mathcal{O}_E$, then $\mathcal{O}_E(K_{A_3+C}) \not\cong \mathcal{O}_E(K_F)$ and we can easily see, as before, that $\operatorname{Im} g_3$ is generated by

$$\operatorname{Im} \{\operatorname{Im} g_2 \otimes H^0(E, K_F) + \operatorname{Im} g_1 \otimes H^0(E, 2K_F) \to \operatorname{Im} g_3\}.$$

If $\mathcal{O}_E(A_1+A_2) \cong \mathcal{O}_E$ then

$$\operatorname{Im} \{\operatorname{Im} g_2 \otimes H^0(E, K_F) + \operatorname{Im} g_1 \otimes H^0(E, 2K_F) \to \operatorname{Im} g_3\}$$

has codimension 1 in $\operatorname{Im} g_3$ and is in fact generated by

$$\operatorname{Im} \{\operatorname{Im} g_1 \otimes R_2 \to \operatorname{Im} g_3\}.$$

But if $x_1 \in \operatorname{Ker} \varphi_1$ and $\beta \in \operatorname{Ker} \varphi_2$ are such that $x_{1|E}$ generates $H^0(E, K_{A_2+A_3+E})$ and that $\beta_{|E} \neq 0$, $\beta \notin \operatorname{Ker} \varphi_1 \otimes R_1$ then, because $\mathcal{O}_E(K_{A_2+A_3+C}) \not\cong \mathcal{O}_E(K_F)$, $x_1\beta$ is a complementary basis of

$\mathrm{Im}\,\{\mathrm{Im}\, g_1 \otimes R_2 \to \mathrm{Im}\, g_3\}$ in $\mathrm{Im}\, g_3$. Hence (A) is proved.

(1.18) **Theorem** Let F be a 1-connected genus 3 fibre, not 2-connected.

(a) If F is analytically of type I then

$R(F, K_F) = k[X_0, X_1, X_2, Y_0, Y_1, Z] / I$.

(b) If F is analytically of type II then

$R(F, K_F) = k[X_0, X_1, X_2, Y_0, Y_1, Y_2, Z_1, Z_2] / I$.

(c) If F is analytically of type III then

$R(F, K_F) = k[X_0, X_1, X_2, Y_0, Y_1, Y_2, Z_0, Z_1, Z_2, Z_3] / I$

where $k[X_0, \ldots]$ is the weighted polynomial ring with $\deg X_i = 1$, $\deg Y_j = 2$, $\deg Z_k = 3$ and none of the relations given by I is linear in one of the variables.

Proof By theorem (II.5.5) $R(F, K_F)$ is generated by its elements of degree ≤ 3. So the question consists of how many generators are needed in degree 2 and 3 .This is answered by (1.2) and (1.4) for degree 2,and (1.10)and (1.17) for degree 3.

Section 2. The canonical ring of a type III fibre.

(2.1) **Theorem** If F is analytically of type III (1.15–1.16) then $R(F, K_F)$ can be presented as

$$R(F, K_F) = k[X_0,X_1,X_2,Y_0,Y_1,Y_2,Z_0,Z_1,Z_2,Z_3] \,/\, M$$

where M is the ideal generated by the 2×2 minors of the following matrices (or table III-1 on next page)

$$A = \begin{pmatrix} \mu X_0 & X_0+\lambda X_1 & X_1 & Y_0 & Y_1 & Z_0 & Z_1 & Z_2 \\ X_0 & X_1 & X_2 & Y_1 & Y_2 & Z_1 & Z_2 & Z_3 \end{pmatrix}$$

$$B = \begin{pmatrix} Z_0-WP & Y_0^2+HQ & Y_0Y_1+WQ & Y_0Y_2+GQ \\ Y_0 & Z_0+WP & Z_1+GP & Z_2+X_1P \end{pmatrix}$$

$$C = \begin{pmatrix} Z_1-GP & Z_3-X_2P & Y_1Y_2+X_1Q & Y_2^2+X_2Q \\ Y_0 & Y_2 & Z_2+X_1P & Z_3+X_2P \end{pmatrix}$$

with

$$W = \lambda^2X_1 + (\lambda+\mu)X_0$$

$$G = \lambda X_1 + X_0$$

$$H = \lambda^3X_1 + (\lambda^2 + \lambda\mu+\mu^3)X_0 \quad \text{and}$$

$$P = \varepsilon X_2^2 + \delta X_1 X_2 + \gamma X_0 X_2$$

$$Q = \alpha_0 X_2 Y_2 + \alpha_1 X_2^3 + \beta_0 X_1 Y_2 + \beta_1 X_1 X_2^2 + \gamma_0 X_0 Y_2 + \gamma_1 X_0 X_2^2$$

In the above, $\lambda = \mu = 0$ if F contains a unique elliptic tail appearing with multiplicity 3, $\lambda = 0$ and $\mu \neq 0$ or $\lambda \neq 0$ and $\mu = 0$ if F contains two distinct tails, one appearing with multiplicity 2, and $\lambda \neq 0$, $\mu \neq 0$ if F contains three distinct elliptic tails.

Remark 1 As a consequence of this theorem ,we can write a non-singular rational parameter space containing all Type III fibres.

Remark 2 The calculations that lead to the proof of (2.1) are quite long (and tedious) but the biggest part is essentially mechanical, using the technique described in (II.1).

Table III-1

The polynomials are presented as the minors of the matrices

$A_{12} : \mu X_0X_1 - X_0(X_0 + \lambda X_1)$

$A_{13} : \mu X_0X_2 - X_0X_1$

$A_{23} : (X_0 + \lambda X_1)X_2 - X_1^2$

$A_{14} : \mu X_0Y_1 - X_0Y_0$

$A_{15} : \mu X_0Y_2 - X_0Y_1$

$A_{24} : (X_0 + \lambda X_1)Y_1 - X_1Y_0$

$A_{25} : (X_0 + \lambda X_1)Y_2 - X_1Y_1$

$A_{34} : X_1Y_1 - X_2Y_0$

$A_{35} : X_1Y_2 - X_2Y_1$

$A_{16} : \mu X_0Z_1 - X_0Z_0$

$A_{17} : \mu X_0Z_2 - X_0Z_1$

$A_{18} : \mu X_0Z_3 - X_0Z_2$

$A_{26} : (X_0 + \lambda X_1)Z_1 - X_1Z_0$

$A_{27} : (X_0 + \lambda X_1)Z_2 - X_1Z_1$

$A_{28} : (X_0 + \lambda X_1)Z_3 - X_1Z_2$

$A_{36} : X_1Z_1 - X_2Z_0$

$A_{37} : X_1Z_2 - X_2Z_1$

$A_{38} : X_1Z_3 - X_2Z_2$

$A_{45} : Y_0Y_2 - Y_1^2$

$A_{46} : Y_0Z_1 - Y_1Z_0$

$A_{47} : Y_0Z_2 - Y_1Z_1$

$A_{48} : Y_0Z_3 - Y_1Z_2$

Table III-1 (continued)

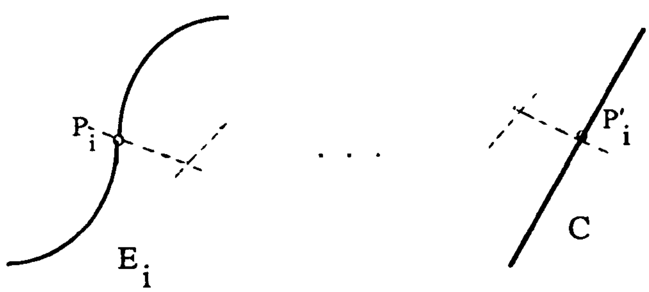
$A_{56} : Y_1Z_1 - Y_2Z_0$
$A_{57} : Y_1Z_2 - Y_2Z_1$
$A_{58} : Y_1Z_3 - Y_2Z_2$
$A_{67} : Z_0Z_2 - Z_1^2$
$A_{68} : Z_0Z_3 - Z_1Z_2$
$A_{78} : Z_1Z_3 - Z_2^2$

$B_{12} : (Z_0-WP)(Z_0+WP) - Y_0(Y_0^2+HQ)$
$B_{13} : (Z_0-WP)(Z_1+GP)-Y_0(Y_0Y_1+WQ)$
$B_{14} : (Z_0-WP)(Z_2+X_1P)-Y_0(Y_0Y_2+GQ)$
$C_{13} : (Z_1-GP)(Z_2+X_1P)-Y_0(Y_1Y_2+X_1Q)$
$C_{14} : (Z_1-GP)(Z_3+X_2P)-Y_0(Y_2^2+X_2Q)$
$C_{23} : (Z_3-X_2P)(Z_2+X_1P)-Y_2(Y_1Y_2+X_1Q)$
$C_{24} : (Z_3-X_2P)(Z_3+X_2P)-Y_2(Y_2^2+X_2Q)$

where $\lambda = \mu = 0$ if $E_1 = E_2 = E_3$, $\lambda = 0$ if $E_2 = E_3$ and E_1 is disjoint from E_2

$W = \lambda^2X_1 + (\lambda+\mu)X_0$
$G = \lambda X_1 + X_0$
$H = \lambda^3X_1 + (\lambda^2+\lambda\mu+\mu^3)X_0$
$P = \varepsilon X_2^2 + \delta X_1X_2 + \gamma X_0X_2$ with $\varepsilon \neq 0$
$Q = \alpha_0X_2Y_2 + \alpha_1X_2^3 + \beta_0X_1Y_2 + \beta_1X_1X_2^2 + \gamma_0X_0Y_2 + \gamma_1X_0X_2^2.$

Proof Let us recall that if F is analytically of type III, then it has a decomposition $F = A_1 + A_2 + A_3 + C$ as in (1.16) where or A_1, A_2, A_3 are disjoint, or $A_1 \supset A_2 \supset A_3$ or $A_i \supset A_j$, $A_k \cap A_i = 0$ for $\{i,j,k\}$ some permutation of $\{1,2,3\}$. If A_i is disjoint from $A_j \cup A_k$ or $A_i = A_3$, then $A_i + C$ is just the elliptic tail $E_i \subset A_i$ linked to C by a simple chain of (-2)-curves (possibly 0) (by II.4.12) and $\mathcal{O}_{E_i}(K_F) \cong \mathcal{O}_{E_i}(A_j + A_k + C - E_i) \cong \mathcal{O}_{E_i}(P_i)$ where $P_i = E_i \cap (A_i + C - E_i)$.



We will assume that the decomposition of F was chosen in such a way that either $A_1 \supset A_2 \supset A_3$ or A_1 is disjoint from $A_2 + A_3$.

Let $R = R(F,K_F)$, $S = R(F-A_1,K_F)$, $T = \operatorname{Im} \varphi$ where $\varphi : S \to R(A_3+C, K_F)$ is the restriction map, and B the image of T under the restriction map $R(A_3 + C, K_F) \to R(C, K_F)$.

By (1.2), (1.9), (1.12) we are in the conditions of (II.1) and we can apply the technique described there to recover successively T from B (step 2), S from T (step 3) and R from S (step 4). B is described in (2.2), T in (2.3) and S in (2.4).

The calculations for T, S and R are essentially mechanical,using (II.1).

Step 1 - The ring B

(2.2) Proposition $B = k[Y_0, Y_1, Y_2, Z_0, Z_1, Z_2, Z_3] / I$ .where I is the ideal generated by the 2×2 minors of the following matrices (or table III-2 on next page).

$$A = \begin{pmatrix} Y_0 & Y_1 & Z_0 & Z_1 & Z_2 \\ Y_1 & Y_2 & Z_1 & Z_2 & Z_3 \end{pmatrix}$$

$$B = \begin{pmatrix} Z_0 & Y_0^2 & Y_0Y_1 & Y_0Y_2 \\ Y_0 & Z_0 & Z_1 & Z_2 \end{pmatrix}$$

$$C = \begin{pmatrix} Z_1 & Z_3 & Y_1Y_2 & Y_2^2 \\ Y_0 & Y_2 & Z_2 & Z_3 \end{pmatrix}$$

Remark In fact I is the ideal generated by the 2x2 minors of the matrix below,but I do not know whether this presentation extends to R.

$$\begin{bmatrix} Y_0 & Y_1 & Z_0 & Z_1 & Z_2 \\ Y_1 & Y_2 & Z_1 & Z_2 & Z_3 \\ Z_0 & Z_1 & Y_0^2 & Y_0Y_1 & Y_0Y_2 \\ Z_1 & Z_2 & Y_0Y_1 & Y_0Y_2 & Y_1Y_2 \\ Z_2 & Z_3 & Y_0Y_2 & Y_1Y_2 & Y_2^2 \end{bmatrix}$$

Table III-2

$R_1{}^4 : Y_1^2 - Y_0Y_2$

$R_1{}^5 : Y_0Z_1 - Y_1Z_0$

$R_2{}^5 : Y_0Z_2 - Y_1Z_1$

$R_3{}^5 : Y_0Z_3 - Y_1Z_2$

$R_4{}^5 : Y_2Z_0 - Y_1Z_1$

$R_5{}^5 : Y_2Z_1 - Y_1Z_2$

$R_6{}^5 : Y_2Z_2 - Y_1Z_3$

$R_1{}^6 : Z_1^2 - Z_0Z_2$

$R_2{}^6 : Z_1Z_2 - Z_0Z_3$

$R_3{}^6 : Z_1Z_3 - Z_2^2$

$R_4{}^6 : Z_0^2 - Y_0^3$

$R_5{}^6 : Z_0Z_1 - Y_0^2Y_1$

$R_6{}^6 : Z_0Z_2 - Y_0^2Y_2$

$R_7{}^6 : Z_1Z_2 - Y_0Y_1Y_2$

$R_8{}^6 : Z_1Z_3 - Y_0Y_2^2$

$R_9{}^6 : Z_2Z_3 - Y_1Y_2^2$

$R_{10}{}^6 : Z_3^2 - Y_2^3$

The superscript indicates the degree of the polynomial.

Proof of (2.2) By (1.12) $B \subset R(C,K_F) = k[X,Y]$ is the subring of $k[X,Y]$ generated by its elements of $\deg \geq 2$. Let $Y_0 = X^2$, $Y_1 = XY$, $Y_2 = Y^2$ and $Z_0 = X^3$, $Z_1 = X^2Y$, $Z_2 = XY^2$, $Z_3 = Y^3$. Then

$$B = k[Y_0, Y_1, Y_2, Z_0, Z_1, Z_2, Z_3] / I \quad \text{where}$$

$k[Y_0, Y_1, Y_2, Z_0, Z_1, Z_2, Z_3]$ is the weighted polynomial ring with $\deg Y_0 = \deg Y_1 = \deg Y_2 = 2$, $\deg Z_0 = \deg Z_1 = \deg Z_2 = \deg Z_3 = 3$ and I is generated by the polynomials in Table III-2.

Let us recall that by (II.4.11) $A_3 + C = E_3 + A + C$ where A is a simple chain of (-2)-curves linking E_3 and C (possibly 0) and that $\mathcal{O}_{E_3}(K_F) \cong \mathcal{O}_{E_3}(P_3)$ where P_3 is the point where E_3 meets A_3-E_3. We can choose X, Y to be such that X in $H^0(C, K_F)$ vanishes at the point P' where C meets A_3-C and such that the only zero in C of Y is at a non-singular point of F.

(End of proof of (2.2).

Step 2 - The ring T .

(2.3) Proposition $T = k[X_2, Y_0, Y_1, Y_2, Z_0, Z_1, Z_2, Z_3] / \tilde{I}$ where $\tilde{I}$ can be described as the ideal generated by the 2x2 minors of the following 3 matrices (or table III-3 on next page).

$$\tilde{A} = \begin{pmatrix} 0 & Y_0 & Y_1 & Z_0 & Z_1 & Z_2 \\ X_2 & Y_1 & Y_2 & Z_1 & Z_2 & Z_3 \end{pmatrix}$$

$$\tilde{B} = \begin{pmatrix} Z_0 & Y_0^2 & Y_0Y_1 & Y_0Y_2 \\ Y_0 & Z_0 & Z_1 & Z_2 \end{pmatrix}$$

$$\tilde{C} = \begin{pmatrix} Z_1 & Z_3-\varepsilon X_2^3 & Y_1Y_2 & Y_2^2+Q \\ Y_0 & Y_2 & Z_2 & Z_3+\varepsilon X_2^3 \end{pmatrix}$$

Table III-3.

$Q_1{}^3 : X_2Y_0$	$R_1{}^6 : Z_1^2-Z_0Z_2$
$Q_2{}^3 : X_2Y_1$	$R_2{}^6 : Z_1Z_2-Z_0Z_3$
$Q_1{}^4 : X_2Z_0$	$R_3{}^6 : Z_1Z_3-Z_2^2$
$Q_2{}^4 : X_2Z_1$	$R_4{}^6 : Z_0^2-Y_0^3$
$Q_3{}^4 : X_2Z_2$	$R_5{}^6 : Z_0Z_1-Y_0^2Y_1$
$R_1{}^4 : Y_1^2-Y_0Y_2$	$R_6{}^6 : Z_0Z_2-Y_0^2Y_2$
$R_1{}^5 : Y_0Z_1-Y_1Z_0$	$R_7{}^6 : Z_1Z_2-Y_0Y_1Y_2$
$R_2{}^5 : Y_0Z_2-Y_1Z_1$	$R_8{}^6 : Z_1Z_3-Y_0Y_2^2$
$R_3{}^5 : Y_0Z_3-Y_1Z_2$	$R_9{}^6 : Z_2Z_3-Y_1Y_2^2$
$R_4{}^5 : Y_2Z_0-Y_1Z_1$	$R_{10}{}^6 : Z_3^2-Y_2^3-Y_2Q-\varepsilon^2X_2^6$
$R_5{}^5 : Y_2Z_1-Y_1Z_2$	
$R_6{}^5 : Y_2Z_2-Y_1Z_3$	

where $Q = \alpha_1 X_2^4 + \alpha_2 X_2^2 Y_2$ and $\varepsilon \neq 0$.

Proof of (2.3) Having already the ring B, by (1.12) we can apply the technique of (II.1) and hence we have $T = k[X_2,Y_0,Y_1,Y_2,Z_0,Z_1,Z_2,Z_3]/\tilde{I} = k[x_2,y_0,y_1,y_2,z_0,z_1,z_2,z_3]$, where the degree of Y_i, Z_i are as before and $\deg X_2 = 1$. To be able to describe $\tilde{I}$ we need to see how the generators of T restrict to $R(E_3,K_F)$.

Since x_2 is the only generator of degree 1 of T, $x_{2|E_3}$ generates $H^0(E_3,K_F)$ and with the choices made for X, Y in step 1, by continuity both y_2 and z_2 do not vanish at the point P_3 such that $\mathcal{O}_{E_3}(K_F) \cong \mathcal{O}_{E_3}(P_3)$. Thus $R(E_3,K_F)$ is generated by the images of x_2, y_2, z_3 and so by (II.1) the ideal generated by x_2 in T is $(x_2) = x_2(k[x_2,y_2] + z_3k[x_2,y_2])$ where $x_2(k[x_2,y_2] + z_3k[x_2,y_2])$ is isomorphic as a k-vector space to $X_2(k[X_2,Y_2] + Z_3 k[X_2,Y_2])$.

Let $J_T = \{t \in T : x_2 t = 0\}$. The generators of J_T as an ideal will be obtained by seeing how y_0, y_1, z_0, z_1, z_2 depend on x_2, y_2, z_3 in $R(E_3,K_F)$. By the choice of X made in step 1 and again by continuity all these elements vanish on P_3 and thus, since $H^0(E_3,2K_F) = \langle x^2_{2|C_3}, y_{2|E_3}\rangle$ and $H^0(E_3,3K_F) = \langle x^3_{2|E_3}, x_2y_{2|E_3}, z_{2|E_3}\rangle$, their restrictions to E_3 will be equal to some multiple of $x_{2|E_3}$.

Making a coordinate change of the y_i's and z_i's by a multiple of x_2 does not affect anything in B or I, and we can assume (making thus a first choice in T) that $y_{0|E_3} = y_{1|E_3} = z_{1|E_3} = z_{2|E_3} = 0$ and so $J_T = (y_0,y_1,z_0,z_1,z_2)$. So (see II.1) $\tilde{I}$ will be generated by

$$Q_1{}^3 : X_2Y_0$$
$$Q_2{}^3 : X_2Y_1$$
$$Q_1{}^4 : X_2Z_0$$
$$Q_2{}^4 : X_2Z_1$$
$$Q_3{}^4 : X_2Z_2$$

and $\tilde{R}_j{}^i = R_j{}^i - X_2 P_j{}^i$, where $R_j{}^i$ are as in Table III-2 of step 1, and $P_j{}^i$ is a polynomial of deg i-1 belonging to $k[X_2, Y_2] + Z_3 k[X_2, Y_2]$.

By the choices made so far $R_j{}^i|_{E_3} \equiv 0$ for all $R_j{}^i$ except $R_{10}{}^6$. Thus $X_2 P_j{}^i \in X_2(K[X_2, Y_2] + Z_3 K[X_2, Y_2]) \cap \tilde{I} = \{0\}$ for $(i,j) \neq (6,10)$, hence $P_j{}^i = 0$ for $(i,j) \neq (6, 10)$.

$\tilde{R}_6{}^{10}$ will be $Z_3^2 - Y_2^3 - X_2 (\alpha_0 X_2^5 + \alpha_1 X_2^3 Y_2 - \alpha_2 X_2 Y_2^2 + \beta_0 Y_2 Z_3 + \beta_1 X_2^2 Z_3)$ and making a new coordinate change by multiples of X_2 $(Z_3 \to Z_3 - \frac{1}{2} \beta_0 X_2 Y_2 - \frac{1}{2} \beta_1 X_2^3)$ we can assume that $\beta_0 = \beta_1 = 0$. Changing also Y_2, if necessary, we can assume that $\alpha_0 \neq 0$.

Thus we have $\tilde{I}$ generated by the polynomials in Table III-3 (to avoid heavy notation we will designate $\tilde{R}_j{}^i$ by $R_j{}^i$).

(End of proof of 2.3.)

Step 3 - The ring S .

(2.4) **Proposition** $S = k[X_1, X_2, Y_0, Y_1, Y_2, Z_0, Z_1, Z_2, Z_3] / L$
where L is the ideal generated by the 2×2 minors of the following matrices (or table III-4 in next page).

$$\begin{pmatrix} \lambda X_1 & X_1 & Y_0 & Y_1 & Z_0 & Z_1 & Z_2 \\ X_1 & X_2 & Y_1 & Y_2 & Z_1 & Z_2 & Z_3 \end{pmatrix}$$

and
$$\begin{pmatrix} Z_0-\lambda^2X_1P & Y_0^2+\lambda^3X_1Q & Y_0Y_1+\lambda^2X_1Q & Y_0Y_2+\lambda X_1Q \\ Y_0 & Z_0+\lambda^2X_1P & Z_1+\lambda X_1P & Z_2+X_1P \end{pmatrix}$$

and
$$\begin{pmatrix} Z_1-\lambda X_1^2P & Z_3-X_2P & Y_1Y_2+X_1Q & Y_2^2+X_2Q \\ Y_0 & Y_2 & Z_2+X_1P & Z_3+X_2P \end{pmatrix}$$

where $\lambda \neq 0$ if $F-A_1$ contains two distinct elliptic tails and $\lambda = 0$ otherwise.

In the above

$$P = \varepsilon X_2^2 + \delta X_1X_2, \text{ with } \varepsilon \neq 0$$

$$Q = \alpha_2X_2Y_2 + \alpha_1X_2^3 + \beta_0X_1Y_2 + \beta_1X_1X_2^2$$

and $\varepsilon^2 = \alpha_0$, $2\varepsilon\delta + \delta^2\lambda = \beta_2$ ($\alpha_0, \alpha_1, \alpha_2$ are the coefficients of F_5, β_0, β_2 the coefficients of G_5 in table III-4).

Table III-4

$Q_1{}^2 : X_1^2 - \lambda X_1 X_2$

$Q_1{}^3 : X_2 Y_0 - X_1 Y_1$

$Q_2{}^3 : X_2 Y_1 - X_1 Y_2$

$Q_3{}^3 : X_1 Y_0 - \lambda X_1 Y_1$

$Q_4{}^3 : X_1 Y_1 - \lambda X_1 Y_2$

$Q_1{}^4 : X_2 Z_0 - X_1 Z_1$

$Q_2{}^4 : X_2 Z_1 - X_1 Z_2$

$Q_3{}^4 : X_2 Z_2 - X_1 Z_3$

$Q_4{}^4 : X_1 Z_0 - \lambda X_1 Z_1$

$Q_5{}^4 : X_1 Z_1 - \lambda X_1 Z_2$

$Q_6{}^4 : X_1 Z_2 - \lambda X_1 Z_3$

$R_1{}^4 : Y_1^2 - Y_0 Y_2$

$R_4{}^5 : Y_0 Z_1 - Y_1 Z_0$

$R_2{}^5 : Y_0 Z_2 - Y_1 Z_1$

$R_3{}^5 : Y_0 Z_3 - Y_1 Z_2$

$R_4{}^5 : Y_2 Z_0 - Y_1 Z_1$

$R_5{}^5 : Y_2 Z_1 - Y_1 Z_2$

$R_6{}^5 : Y_2 Z_2 - Y_1 Z_3$

$R_1{}^6 : Z_1^2 - Z_0Z_2$

$R_2{}^6 : Z_1Z_2 - Z_0Z_3$

$R_3{}^6 : Z_1Z_3 - Z_2^2$

$R_4{}^6 : Z_0^2 - Y_0^3 - \lambda^5 X_1(F_5 + \lambda G_5)$

$R_5{}^6 : Z_0Z_1 - Y_0^2Y_1 - \lambda^4 X_1(F_5 + \lambda G_5)$

$R_6{}^6 : Z_0Z_2 - Y_0^2Y_2 - \lambda^3 X_1(F_5 + \lambda G_5)$

$R_7{}^6 : Z_1Z_2 - Y_0Y_1Y_2 - \lambda^2 X_1(F_5 + \lambda G_5)$

$R_8{}^6 : Z_1Z_3 - Y_0Y_2^2 - \lambda X_1(F_5 + \lambda G_5)$

$R_9{}^6 : Z_2Z_3 - Y_1Y_2^2 - X_1(F_5 + \lambda G_5)$

$R_{10}{}^6 : Z_3^2 - Y_2^3 - X_2F_5 - X_1G_5$

where

$F_5 = \alpha_2X_2Y_2^2 + \alpha_1X_2^3Y_2 + \alpha_0X_2^5$ with $\alpha_0 \neq 0$

$G_5 = \beta_0X_2Y_2^2 + \beta_1X_2^3Y_2 + \beta_2X_2^5$

and $\lambda = 0$ if $E_2 = E_3$.

Proof of (2.4) We have $S = R(A_2 + A_3 + C_1K_F)$. By (1.9) we can again apply the technique of (II.1), as in step 2, to recover S from T. Let E_2 be the elliptic tail contained in A_2 .

Then $S = k[X_1,X_2,Y_0,Y_1,Y_2,Z_0,Z_1,Z_2,Z_3] / L$ where $x_1 \in S_1$ is such that $x_{1|A_2+A_3+C-E_2} = 0$.

Let $J_S = \{s \in S : x_1 s = 0\}$. As in step 2 to determine generators for J_S we will have to find out how $x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2, z_3 \in S$ restrict to $R(E_2,K_F)$.

We have two cases to consider: A_2 disjoint from A_3 and $A_2 \supset A_3$ and $E_2 = E_3$. We will treat them separately. Although it is possible to do them at the same time using syzygies, the calculations are in fact lengthier.

Proof of (2.4) Case (a) $E_2 = E_3$

This is a harder case since we have to look at the nilpotent structure of x_1 and so use syzygies. If $E_2 = E_3$ the restriction map $S \to R(E_2,K_F)$ factors through T and thus $x_{2|E_2}$, $y_{2|E_2}$, $z_{3|E_2}$ generate $R(E_2,K_F)$ and (by the choices made in step 2) $y_{0|E_2} = y_{1|E_2} = z_{0|E_2} = z_{2|E_2} = 0$.

Since $x_{1|A_2+A_3+C-E_2} = 0$ and thus $x_{1|E_2} = 0$, we have

$J_S = (x_1,y_0,y_1,z_0,z_2)$ and $(x_1) = x_1(k[x_2,y_2] + z_3\, k[x_2,y_2])$

(which is isomorphic as a k-vector space to $X_1(k[X_2,Y_2] + Z_3\, k[X_2,Y_2])$

So L is generated by $X_1^2, X_1Y_0, X_1Y_1, X_1Z_0, X_1Z_1, X_1Z_2$ and $\tilde{Q}_j^{\,i} = Q_j^{\,i} - X_1 P_j^{\,i}$, $\tilde{R}_j^{\,i} = F_j^{\,i} - X_1 N_j^{\,i}$ where $N_j^{\,i}$, $P_j^{\,i}$ are polynomials of degree i-1 belonging to $k[X_2,Y_2] + Z_3k[X_2,Y_2]$ and $Q_j^{\,i}$, $R_j^{\,i}$ are as in Table III-3.

Since $E_2 = E_3$ and $x_{1|E_2} = 0$ we can not determine an exact form for the $\tilde{R}_j^{\,i}$, $\tilde{Q}_j^{\,i}$ by considering their restrictions to E_2 as in step 2 and we will have to look at the syzygies between the $R_j^{\,i}$ and $Q_j^{\,i}$ to find $P_j^{\,i}$, $N_j^{\,i}$.

Now $\tilde{Q}_j^{\,i} = Q_j^{\,i} - X_1 P_j^{\,i}$ with $\deg P_j^{\,i} = i-1$. So $\tilde{Q}_1^{\,3}$ is $X_2Y_0 - \varepsilon_1X_1Y_2 - \varepsilon_0X_1X_2^2$. Since a coordinate change by a multiple of X_1 does not change anything done so far we can change Y_0 so that $Q_1^{\,3}$ is $X_2Y_0 - \varepsilon_1X_1Y_2$.

Similarly we can change Y_1, Z_0, Z_1, Z_2 so that

$\tilde{Q}_2^{\,3} = X_2Y_1 - \varepsilon_2X_1Y_2$

$\tilde{Q}_1^{\,4} = X_2Z_0 - \gamma_0X_1Z_2$

$\tilde{Q}_2^{\,4} = X_2Z_1 - \gamma_1X_1Z_2$

$\tilde{Q}_3^{\,4} = X_2Z_2 - \gamma_2X_1Z_3$.

We can obtain some of the syzygies between the elements $\tilde{I}$, by considering the 3×3 minors of the matrices obtained from $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ by doubling one of the rows.

From the syzygy between elements of $\tilde{I}$, $Y_1\, Q_2^{\,3} - X_2\, R_1^{\,4} - Y_2\, Q_1^{\,3}$ we get $-X_1Y_1\, P_2^{\,3} + X_1X_2\, N_1^{\,4} + X_1Y_2\, P_1^{\,3} \in L$. Since $X_1Y_1 \in L$ we then have $X_1X_2\, N_1^{\,4} + X_1Y_2\, P_1^{\,3} \in L \cap X_1(k[X_2,Y_2] + Z_3\, k[X_2,Y_2]) = \{0\}$ and

thus $X_1X_2\, N_1^{\,4} = -X_1Y_2\, P_1^{\,3}$. By the choice made above $P_1^{\,3} = \varepsilon_1X_1\, Y_2^2$. Since $N_1^{\,4}$ is a polynomial of deg 3 in X_2, Y_2, Z_2 necessarily $P_1^{\,3} = N_1^{\,4} = 0$.

In the same way considering the other syzygies arising from the matrix $\tilde{A}$ we obtain $\gamma_0 = \gamma_1 = 0$, $\varepsilon_2 = \gamma_2$, and $N_j^{\,i} = 0$ for all $i < 6$ and $N_1^{\,6} = N_2^{\,6} = N_3^{\,6} = 0$.

The other polynomials can be obtained easily by considering the following syzygies:

$$X_2 R_9{}^6 - Z_3 Q_3{}^4 + Y_2^2 Q_2{}^3$$

$$X_2 R_8{}^6 - Z_3 Q_2{}^4 + Y_2^2 Q_1{}^3$$

$$X_2 R_7{}^6 - Z_2 Q_2{}^4 + Y_0 Y_2 Q_2{}^3$$

$$X_2 R_6{}^6 - Z_2 Q_1{}^4 + Y_0 Y_2 Q_1{}^3$$

$$X_2 R_5{}^6 - Z_0 Q_2{}^4 + Y_0^2 Q_2{}^3$$

$$X_2 R_4{}^6 - Z_0 Q_1{}^4 + Y_0^2 Q_1{}^3$$

giving

$N_9{}^6 = \varepsilon_2 Y_2(\alpha_0 X_2^3 + \alpha_1 X_2 Y_2) + \varepsilon_2 \varepsilon^2 X_2^5$ (with $\alpha_0, \alpha_1, \varepsilon$ as in Table III-3)

$N_8{}^6 = N_7{}^6 = N_6{}^6 = N_5{}^6 = N_4{}^6 = 0$.

As an example of how these identities were obtained, consider the first syzygy. $\tilde{Q}_3{}^4$ is $X_2Z_2 - \varepsilon_2 X_1 Z_3$ and $\tilde{Q}_2{}^3$ is $X_2Y_1 - \varepsilon_2 X_1 Y_2$ and so from the first syzygy we have $-X_2X_1N_9{}^6 + \varepsilon_2 X_1 Z_3^2 - \varepsilon_2 X_1 Y_2^3 \in L$ and thus

$-X_2X_1\,N_9{}^6 + \varepsilon_2 X_1 Z - \varepsilon_2 X_1 \tilde{R}_{10}{}^6 = -X_1X_2N_9{}^6 + \varepsilon_2 X_1 X_2 Y_2(\alpha_0 X_2^3 + \alpha_1 X_2 Y_2) + \varepsilon_2 \varepsilon^2 X_1 X_2^6 + \varepsilon_2 X_1^2 N_{10}{}^6 \in L$.

Then since $X_1^2 \in L$, $-X_1X_2N_9{}^6 + \varepsilon_2 X_1 X_2 Y_2 (\alpha_0 X_2^3 + \alpha_1 X_2 Y_2) + \varepsilon_2 \varepsilon^2 X_1 X_2^6 \in L \cap X_1 (k[X_2,Y_2] + Z_3\, k[X_2,Y_2]) = \{0\}$ and hence

$N_9{}^6 = \varepsilon_2 Y_2 (\alpha_0 X_2^3 + \alpha_1 X_2 Y_2) + \varepsilon_2 \varepsilon^2 X_2^5 .$

Let us remark that since

$$Q_1{}^3 = X_2 Y_0 \quad \text{and}$$
$$\tilde{Q}_2{}^3 = X_2 Y_1 - \varepsilon_2 X_1 Y_2$$

ε_2 must be non zero. Otherwise both y_0 and y_1 would belong to the kernel of $S_2 \to H^0(2E, 2K_F)$ which is 1-dimensional.

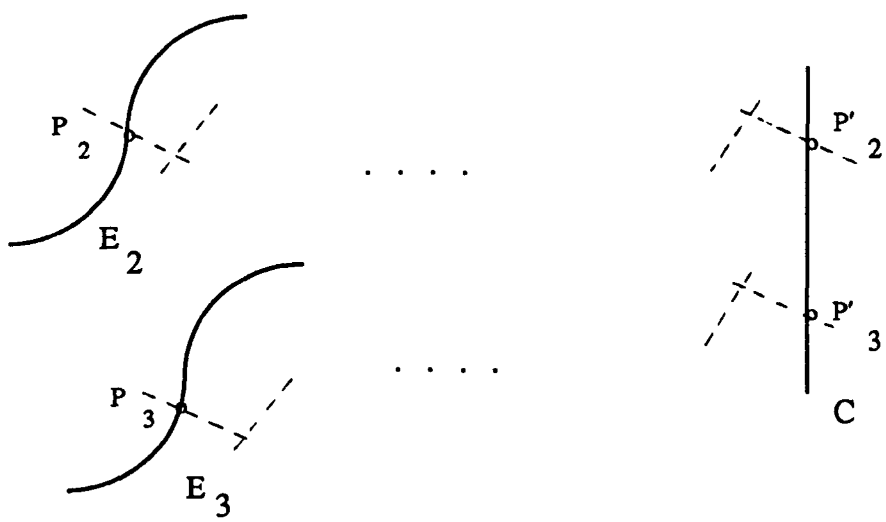
Changing X_1 by a suitable multiple of X_1 we can assume that $\varepsilon_2 = 1$. We will, as before, assume that (by an appropriate coordinate change) $N_{10}{}^6$ is

$\beta_0 X_2 Y_2^2 + \beta_1 X_2^3 Y_2 + \beta_2 X_2^5 .$

The ideal L is then described in Table III-4 with $\lambda = 0$.

Proof of (2.4) Case (b) $E_2 \neq E_3$

If $E_2 \neq E_3$ then A_2 is disjoint from A_3 and A_2-E_2 and E_2 have no common components. So $A_2 + A_3 + C$ looks like



where the doted lines are chains of (-2) curves, possibly 0.

Since $x_{1|A_2+A_3+C-E_2} = 0$ necessarily $x_{1|E_2}$ generates $H^0(E_2,K_F)$. Let $P_2 = E_2 \cap (A_2-E_2)$. Then $\mathcal{O}_{E_2}(K_F) \cong \mathcal{O}_{E_2}(P_2)$ (because $\mathcal{O}_{E_2}(A_1) \cong \mathcal{O}_{E_2}$) and thus by continuity and the choice of Y in step 1 (the unique zero of y in C was at a non-singular point of F) and the choice of X (the unique zero of x in C was at the point where A_3 meets C) we have that none of the y_i's or z_i's vanish on P_2.

So $x_{1|E_2}, y_{2|E_2}, z_{3|E_2}$ generate $R(E_2,K_F)$, y_0, y_1 restricted to E_2 are equal to some linear combination of $x_1{}^2{}_{|E_2}$, $y_{2|E_2}$ where y_2 appears with non-zero coefficient and z_0, z_1, z_2 restricted to E_2 are equal to some linear combination of x_1^3, x_1y_2, z_3 where z_3 appears also with a nonzero coefficient.

We can change, as before, the Y_i's and Z_i's by multiples of X_1 and assume that

$y_{0|E_2} = \lambda_0 y_{2|E_2}$, $y_{1|E_2} = \lambda_1 y_{2|E_2}$, $z_{0|E_2} = \mu_0 z_3$, $z_{1|E_2} = \mu_1 z_{3|E_2}$, $z_{2|E_2} = \mu_2 z_{3|E_2}$.

We have also $x_{2|E_2} = \gamma x_{1|E_2}$ for some $\gamma \in k$.

We could now use the syzygies between the elements of $\tilde{I}$ as in case (a) to deduce the relations between the λ_i's and the exact form of $\tilde{R}_i{}^j$, $\tilde{Q}_k{}^p$ but it is easier to proceed as in step 2 and consider the images of these polynomials in $R(E_2,K_F)$.

For example consider $\tilde{R}_1{}^4 = R_1{}^4 - X_1P_1{}^4 = Y_1^2 - Y_0Y_2 - X_1P_1{}^4$ where $P_1{}^4 \in k[X_1, Y_2, Z_3]_3$. Then in $R(E_2,K_F)$ $y_1^2 - y_0y_2 - x_1p_1{}^4 = 0$ and so, since $y_{1|E_2} = \lambda_1 y_{2|E_2}$ and $y_{0|E_2} = \lambda_0 y_{2|E_2}$, we get $(\lambda_1^2 - \lambda_0)y_2^2 - x_1 P_1{}^4 = 0$ in $R(E_2,K_F)$ .Thus $\lambda_1^2 = \lambda_0$ and $P_1{}^4 = 0$.

Using the same reasoning (with $R_3{}^5$, $R_2{}^5$, $R_1{}^5$ in $\tilde{I}$) we obtain $\mu_2 = \lambda_1$, $\mu_1 = \lambda_1^2$, $\mu_0 = \lambda_1^3$.

Since adding to X_2 a scalar multiple of X_1 does not affect T or $\tilde{I}$ or the choices made so far in this step, we will change X_2 so that $x_{1|E_2} = \lambda_1 x_{2|E_2}$. This choice is made in order to give a similar presentation to S in both cases (a) and (b). Denoting λ by λ_1 we then have

$$J_S = (x_1-\lambda x_2, y_0-\lambda^2 y_2, y_1-\lambda y_2, z_0-\lambda^3 z_3, z_1-\lambda^2 z_3, z_2-\lambda z_3)$$

with $\lambda \neq 0$ and (x_1) in S is equal to $x_1(k[x_2,y_2] + z_3\, k[x_2,y_2])$ and isomorphic as a k-vector space to $X_1(k[X_2,Y_2] + Z_3\, k[X_2,Y_2])$.

We will then have $\tilde{R}_i{}^j = R_i{}^j - X_1 P_i{}^j$ and $\tilde{Q}_k{}^\ell = Q_k{}^\ell - X_1 P_k{}^\ell$, with $P_m{}^n \in (k[X_2,Y_2] + Z_3\, k[X_2,Y_2])_{n-1}$.

Let $\tilde{Q}_1{}^3$ be $X_2Y_0 - \varepsilon_0 X_1 Y_2 - \varepsilon_1 X_1 X_2^2$. Then in $R(F_2,K_F)$ we have $x_2 y_0 = \varepsilon_0 x_1 y_2 + \varepsilon_1 x_1 x_2^2$ and this equality reads (because of our previous choices) $\frac{1}{\lambda}\lambda^2 x_1 y_2 = \varepsilon_0 x_1 y_2 + \frac{\varepsilon_1}{\lambda^2} x_1^3$ and thus $\varepsilon_0 = \lambda$, $\varepsilon_1 = 0$.

Doing the same for $\tilde{Q}_2{}^3$, $\tilde{Q}_1{}^4$, $\tilde{Q}_2{}^4$, $\tilde{Q}_3{}^4$, $\tilde{R}_1{}^4$, ... , $\tilde{R}_6{}^5$, $\tilde{R}_1{}^6$, $\tilde{R}_2{}^6$, $\tilde{R}_3{}^6$ we obtain

$\tilde{Q}_2{}^3 : X_2Y_1 - X_1Y_2$

$\tilde{Q}_1{}^4 : X_2Z_0 - \lambda^2 X_1 Z_3$

$\tilde{Q}_2{}^4 : X_2Z_1 - \lambda X_1 Z_3$

$\tilde{Q}_3{}^4 : X_2Z_2 - X_1Z_3$

and $\tilde{R}_i{}^k = R_i{}^k$ for all $k < 6$ and $k = 6$, $i = 1, 2$ or 3.

$\tilde{R}_{10}{}^6$ will be of the form $Z_3^2 - Y_2^3 - X_2F_5 - X_1\,P_{10}{}^6$ (and is the equation describing E_2). We can change Z_3 (and Z_2, Z_1, Z_0 accordingly so that the previous polynomials remain unchanged) to have $P_{10}{}^6 \in k[X_2,Y_2]_5$ (i.e. without terms in Z_3).

From $\tilde{R}_{10}{}^6$ we can obtain $\tilde{R}_4{}^6$,..., $\tilde{R}_9{}^6$. As an example consider

$\tilde{R}_4{}^6 : Z_0^2 - Y_0^3 - X_1\,P_4{}^6$, $P_4{}^6 \in K[X_2, Y_2, Z_3]_5$.

Restricting down to E_2 we get

$$\lambda^6\, z_2^2 - \lambda^6\, y_2^3 - \lambda\, x_2 - p_4{}^6 = 0$$

and thus since F_5, $P_{10}{}^6$, $P_4{}^6 \in k[X_2, Y_2, Z_3]_5$ this relation only holds in $R(E_2,K_E)$ if $P_4{}^6 = \lambda^5\,(F_5 + \lambda P_{10}{}^6)$. In the same way, denoting $P_{10}{}^6$ by G_5, we obtain:

$\tilde{R}_4{}^6 : Z_0^2 - Y_0^3 - \lambda^5\,X_1\,(F_5 + \lambda G_5)$

$\tilde{R}_5{}^6 : Z_0Z_1 - Y_0^2Y_1 - \lambda^4\,X_1\,(F_5 + \lambda G_5)$

$\tilde{R}_6{}^6 : Z_0Z_2 - Y_0^2Y_2 - \lambda^3\,X_1\,(F_5 + \lambda G_5)$

$\tilde{R}_7{}^6 : Z_1Z_2 - Y_0Y_1Y_2 - \lambda^2\,X_1\,(F_5 + \lambda G_5)$

$\tilde{R}_8{}^6 : Z_1Z_3 - Y_0Y_2^2 - \lambda\,X_1\,(F_5 + \lambda G_5)$

$\tilde{R}_9{}^6 : Z_2Z_3 - Y_1Y_2^2 - X_1\,(F_5 + \lambda G_5)$

$\tilde{R}_{10}{}^6 : Z_3^2 - Y_2^3 - X_2F_5 - X_1G_5$

Then we have in both cases L given by Table III-4 (we drop again $\sim$), where $\lambda \neq 0$ if and only if $E_2 \neq E_3$. (End of proof of 2.4.)

Step 4 - The ring R (2.1)

We have $R = \oplus R_n = \oplus H^0(F, nK_F)$. Let E_1 be the elliptic tail contained in A_1. By proposition (1.2) we can again apply the technique of (II.1) as in steps 2 and 3 and we will skip the details where the reasoning is the same.

So $R = k[X_0, X_1, X_2, Y_0, Y_1, Y_2, Z_0, Z_1, Z_2, Z_3] / M$

where $x_0 \in R$ is such that $x_0 \in R_1$ and $x_{0|F-E_1} \equiv 0$.

M will be generated by pull-backs of the generators of L (giving S) and the polynomials C_i such that the ideal $J_R = \{s \in R : x_0 s = 0\}$ is generated by c_i. The ideal J_R will be generated by two homogeneous elements of degree 1, two homogeneous elements of degree 2 and three homogeneous elements of degree 3, obtained by seeing how the x_i's, y_i's and z_i's restrict to $R(E_1, K_F)$.

Either $E_1 \subset F\text{-}A_1$ or E_1 is not contained in $F\text{-}A_1$. We can assume that if $E_1 \subset F\text{-}A_1$ then $E_1 = E_2 = E_3$ (and so in S, $\lambda = 0$). If $E_1 \subset F\text{-}A_1$ and $E_2 \neq E_3$ we can always choose another decomposition of F such that A'_1 and $F\text{-}A'_1$ have no common components.

We will deal with these two cases separately.

Proof of (2.1) Case (a) $E_1 \subset F\text{-}A_1$

By our assumption $E_1 = E_2 = E_3$. Then S, given in Table III-4 (step 3) has $\lambda = 0$.

As in step 3 case (a) we have by our previous assumptions that $x_{2|E_1}$, $y_{2|E_1}$, $z_{2|E_1}$ generate $R(E_1, K_E)$ and $J_R = (x_0, x_1, y_0, y_1, z_0, z_1, z_3)$ Thus the polynomials X_0C_i will be

$Q_2^{\,2} : X_0^2$ $\quad$ $Q_5^{\,3} : X_0Y_0$ $\quad$ $Q_7^{\,4} : X_0Z_0$ $\quad$ $Q_9^{\,4} : X_0Z_2$

$Q_2^{\,3} : X_0X_1$ $\quad$ $Q_6^{\,3} : X_0Y_1$ $\quad$ $Q_8^{\,4} : X_0Z_1$

and (x_0) in R will be equal to $x_0\,(k[x_2,y_2] + z_3\,k[x_2,y_2])$ and isomorphic to $X_0(k[X_2, Y_2] + Z_3\,k[X_2, Y_2])$ as a k-vector space.

To determine the exact form of the pull-backs of elements of L we can proceed exactly as in step 3 case (a) and we omit the calculations since they are mechanical and repetitive.

We make a coordinate change so that

$\tilde{Q}_1^{\,3}$ becomes $X_2Y_0 - X_1Y_1 - \gamma_1X_0Y_2$
$\tilde{Q}_2^{\,3}$ becomes $X_2Y_1 - X_1Y_2 - \gamma_2X_0Y_2$
$\tilde{Q}_1^{\,4}$ becomes $X_2Z_0 - X_1Z_1 - \gamma_3X_0Z_3$
$\tilde{Q}_2^{\,4}$ becomes $X_2Z_1 - X_1Z_2 - \gamma_4X_0Z_3$
$\tilde{Q}_2^{\,5}$ becomes $X_2Z_2 - X_1Z_3 - \gamma_5X_0Z_3$.

Now $\tilde{Q}_1^{\,2}$ is $X_1^2 - \gamma X_0X_2$ for some γ.

Using the syzygies as in step 3 case (a) we get $\tilde{Q}_1^{\,3} = Q_1^{\,3}$, $\tilde{Q}_3^{\,3} = Q_3^{\,3}$, $\tilde{Q}_1^{\,4} = Q_1^{\,4}$, $\tilde{Q}_2^{\,4} = Q_2^{\,4} = \tilde{Q}_4^{\,4} = Q_4^{\,4}$, $\tilde{Q}_5^{\,4} = Q_5^{\,4}$, and $\tilde{R}_i^{\,k} = R_i^{\,k}$ for all $k = 4, 5$ and for $k = 6$, $i = 1, 2$ or 3.

We also get

$\tilde{Q}_2{}^3 : X_2Y_1 - X_1Y_2 - \varepsilon X_0Y_2$
$\tilde{Q}_4{}^3 : X_1Y_1 - \gamma X_0Y_2$
$\tilde{Q}_3{}^4 : X_2Z_2 - X_1Z_3 - \varepsilon X_0Z_3$
$\tilde{Q}_6{}^4 : X_1Z_2 - \gamma X_0Z_3$.

We can then make a coordinate change $X_1 \to (X_1 - \varepsilon_0 X_0)$ so that $\tilde{Q}_2{}^3$ becomes equal to $Q_2{}^3$ and $\tilde{Q}_3{}^4$ becomes equal to $Q_3{}^4$. This, as it is easily checked, does not alter the other polynomials.

Let $\tilde{R}_{10}{}^6$ be $Z_3^2 - Y_2^3 - X_2F_5 - X_1G_5 - X_0H_5$ with $H_5 \in k[X_2,Y_2,Z_3]_5$. We can change Z_3 by a multiple of X_0 so that $H_5 \in k[X_2,Y_2]_5$. It is easy to check that this does not change the aspect of the other polynomials.

Then using the syzygies again we get

$\tilde{R}_6{}^9 : Z_2Z_3 - Y_1Y_2^2 - X_1F_5 - \gamma X_0G_5$

$\tilde{R}_6{}^8 : Z_1Z_3 - Y_0Y_2^2 - \gamma X_0F_5$

and $\tilde{R}_4{}^6 = R_4{}^6$, $\tilde{R}_5{}^6 = R_5{}^6$, $\tilde{R}_6{}^6 = R_6{}^6$, $\tilde{R}_7{}^6 = R_7{}^6$.

Now γ must be different from zero. Otherwise $x_1^2 = 0$ in R which is impossible since x_1 vanishes only once in E . Changing X_0 by a scalar multiple we can assume that $\gamma = 1$.

We then obtain that M is the ideal as in (2.1) with $\lambda = \mu = 0$.

Proof of 2.1. Case (b) $E_1 \not\subset F-A_1$

In these conditions A_1 has no common components with $F-A_1$ and so A_1 is disjoint from A_2 and A_3 (see picture (II.4.15)).

We can use exactly the same reasoning as in step 3 case (b) to see that $R(E_1, K_F)$ is generated by $x_{0|E_1}$, $y_{2|E_1}$, $z_{3|E_1}$ and that (using a suitable choice of coordinates)

$y_{0|E_2} = \mu^2 y_{2|E_2}$, $y_{1|E_2} = \mu y_{2|E_2}$, $z_{0|E_2} = \mu^3 z_{3|E_3}$, $z_{1|E_2} = \mu^2 z_{3|E_2}$ and $z_{2|E_2} = \mu z_{3|E2}$, with $\mu \neq 0$.

Let us remark that if $\lambda \neq 0$ (i.e. $E_2 \neq E_3$) then $\lambda \neq \mu$. Otherwise the points P_1 and P_2 where A_1 and A_2 meet C would coincide.

As in step 3 (b) we will change X_2 and X_1 by multiples of X_0 so that $x_{0|E_1} = \mu(\mu-\lambda)x_{2|E_1}$ and $x_{1|E_2} = \mu x_{2|E_2}$ (i.e. $x_{0|E_1} = \mu-\lambda)x_{1|E_1}$) . This choice is made in order to give a similar presentation to the ring R in cases (a) and (b).Then we have:

$J_R = (x_0 - \mu(\mu-\lambda)x_2 , x_1 - \mu x_2, y_0-\mu^2 y_2, y_1-\mu y_2 , z_0-\mu^3 z_3, z_1-\mu^2 z_3, z_2-\mu z_3)$
and proceeding as in step 3(b) we will determine $\tilde{Q}_k{}^{\ell}$ and $\tilde{R}_i^j$ by looking at the restrictions of $Q_k{}^{\ell}$ and R_i^j to $R(E_1,K_F)$.

With the choices made so far we have that (x_0) in R is equal to $x_0(k[x_2,y_2] + z_3\, k[x_2, y_2])$ which is isomorphic to $X_0(k[X_2,Y_2] + Z_3\, k[X_2,Y_2])$.

We will omit all the calculations now since they are similar to those in step 2 and step 3 (b). We get, because of the previous choice of coordinates:

$\tilde{Q}_1{}^2 : X_1^2 - \lambda X_1X_2 - X_0X_2$
$\tilde{Q}_1{}^3 : X_2Y_0 - X_1Y_1$
$\tilde{Q}_2{}^3 : X_2Y_1 - X_1Y_2$
$\tilde{Q}_3{}^3 : X_1Y_0 - \lambda X_1Y_1 - \mu X_0Y_2$
$\tilde{Q}_4{}^3 : X_1Y_1 - \lambda X_1Y_2 - X_0Y_2$
$\tilde{Q}_1{}^4 : X_2Z_0 - X_1Z_1$
$\tilde{Q}_2{}^4 : X_2Z_1 - X_1Z_2$

$\tilde{Q}_3{}^4 : X_2Z_2 - X_1Z_3$
$\tilde{Q}_4{}^4 : X_1Z_0 - \lambda X_1Z_1 + \mu^2 X_0Z_3$
$\tilde{Q} : X_1Z_1 - \lambda X_1Z_2 - \mu X_0Z_3$
$\tilde{Q}_6{}^4 : X_1Z_2 - \lambda X_1Z_3 - X_0Z_3$
and then $\tilde{R}_i{}^k = R_i{}^k$ for $k = 4, 5$, $\tilde{R}_1{}^6 = R_1{}^6$, $\tilde{R}_2{}^6 = R_2{}^6$, $\tilde{R}_3{}^6 = R_3{}^6$. If (with an appropriate change of coordinates)

$$\tilde{R}_{10}{}^6 \text{ is } Z_3^2 - Y_2^3 - X_2^2F_4 - X_1X_2G_4 - X_0X_2H_4$$

where $F_4, G_4, H_4 \in k[X_2,Y_2]_4$ and $X_2F_4 = F_5$, $X_2G_4 = G_5$ of Table III-4, we have, letting $N = X_2F_4 + X_1G_4 + X_0H_4$:

$\tilde{R}_9{}^6 : Z_2Z_3 - Y_1Y_2^2 - X_1\,N$

$\tilde{R}_8{}^6 : Z_1Z_3 - Y_0Y_2^2 - (X_0 + \lambda X_1)\,N$

$\tilde{R}_7{}^6 : Z_1Z_2 - Y_0^2Y_2 - \left((\lambda+\mu)X_0 + \lambda^2X_1\right)N$
$\tilde{R}_6{}^6 : Z_0Z_2 - Y_0Y_1Y_2 - \left((\lambda^2+\lambda\mu+\mu^2)X_0 + \lambda^3X_1\right)N$

$\tilde{R}_5{}^6 : Z_0Z_1 - Y_0^2Y_1 - \left((\lambda^3 + \lambda^2\mu + \lambda\mu^2 + \mu^3)X_0 + \lambda^4X_1)\right)N$

$\tilde{R}_4{}^6 : Z_0^2 - Y_0^3 - \left((\lambda^4+\lambda^3\mu+\lambda^2\mu+\lambda\mu^2+\lambda\mu^3+\mu^4)X_0 + \lambda^5)\right)N$.

Letting $H_4 = \gamma_0Y_2^2 + \gamma_1X_2^2Y_2 + \gamma_2X_2^4$ we can rewrite the polynomials above (using the previous relations) as:

$(Z_3-X_2P)(Z_3+X_2P) - Y_2(Y_2^2+X_2Q)$

$(Z_3-X_2P)(Z_2+X_1P) - Y_2(Y_1Y_2+X_1Q)$

$$(Z_3+X_2P)(Z_1-(\lambda X_1+X_0)P) - Y_0(Y_2^2+X_2Q)$$

$$(Z_1-(\lambda X_1+X_0)P)(Z_2+X_1P) - y_0(Y_1Y_2+X_1Q)$$

$$(Z_0-(\lambda^2X_1+(\lambda+\mu)X_0P)(Z_2+X_1P)-Y_0(Y_0Y_2+(\lambda X_1+X_0)Q)$$

$$(Z_0-(\lambda^2X_1+(\lambda+\mu)X_0)P)(Z_1+(\lambda X_1+X_0)P)-Y_0(Y_0Y_1+(\lambda^2X_1+(\lambda+\mu)X_0)Q)$$

$$(Z_0-(\lambda^2X_1+(\lambda+\mu)X_0)P)(Z_0+(\lambda^2X_1+(\lambda+\mu)X_0P)-$$
$$Y_0(Y_0^2+(\lambda^3X_1(\lambda^2+\lambda\mu+\mu^2))Q)$$

where

$$P = \varepsilon X_2^2 + \delta X_1X_2 + \gamma X_0X_2\,, \text{ with } \gamma^2=\gamma_2.$$

$$Q = \alpha_0X_2Y_2 + \alpha_1X_2^3 + \beta_0X_1Y_2 + \beta_1X_1X_2^2 + \gamma_0X_0Y_2 + \gamma_1X_0X_2^2\,.$$

In Table III-1 we have the ideal M for both cases (a) (b).

End of proof of (2.1)

Section 3. The canonical ring of a type II fibre.

(3.1) Theorem If F is analytically of type II (1.13–1.14) then $R(F, K_F)$ can be presented as $k[X_0, X_1, X_2, Y_0, Y_1, Y_2, Z_1, Z_2] / M$, where M is an ideal generated by 20 polynomials (3 of degree 2, 4 of degree 3, 6 of degree 4, 4 of degree 5, 3 of degree 6) and

(i) If a standard decomposition of F, $F = \sum_{i=1}^{n} A_i + D$ has $n = 2$, M is the ideal generated by the 2×2 minors of the matrices

$$\begin{pmatrix} 0 & X_2 & Y_0 & Y_2+\gamma Y_1 & Z_2+\gamma Z_1 \\ X_0 & Y_2-\gamma Y_1 & Z_2-\gamma Z_1 & P & P' \end{pmatrix}$$

$$\begin{pmatrix} -2\gamma X_0 & X_0 & Y_2-\gamma Y_1 & Z_2-\gamma Z_1 \\ X_0 & X_1 & Y_1 & Z_1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & X_2 & Y_0 \\ X_1 & Y_0 & X_2Y_1 \end{pmatrix}$$

$$\begin{pmatrix} X_2 & Y_1 & Z_1+\varepsilon_0X_1^3+\varepsilon_1X_0X_1^2 & (Z_2-\gamma Z_1)+\varepsilon_0X_0X_1^2+\varepsilon_1X_0^2X_1 \\ Y_0 & Z_1-\varepsilon_0X_1^3-\varepsilon_1X_0X_1^2 & Y_1^2+X_1Q & Y_1(Y_2-\gamma Y_1)+X_0Q \end{pmatrix}$$

where

$$P = \alpha_0 X_2{}^3 + \alpha_1 X_2 Y_0 + \alpha_2 X_2 Y_1 + \alpha_3 Z_1$$
$$P' = \alpha_0 X_2{}^2 Y_0 + \alpha_1 Y_0{}^2 + \alpha_2 Y_0 Y_1 + \alpha_3 Y_1{}^2$$
$$Q = \beta_0 X_1{}^3 + \beta_1 X_0 X_1{}^2 + \beta_2 X_1{}^2 Y_1 + \beta_3 X_0 X_1 Y_1 \ .$$

In the above $\gamma \neq 0$, if A_1 and A_2 are disjoint, and $\gamma = 0$ if $A_1 \supset A_2$. (For an explicit list of a set of generators for M see (3.3)).

(ii) If a standard decomposition of F, $F = \sum_{i=1}^{n} A_i + D$ has $n = 3$ or has $n = 2$ and A_1, A_2 disjoint, then M is the ideal generated by the 2×2 minors of the matrices

$$\begin{pmatrix} 0 & X_1 - \lambda^2 X_0 & X_0 + \lambda X_2 & Y_0 & Y_1 - \lambda^2 Y_2 - \lambda^2 \varepsilon_0 Y_0 & Z_1 - \lambda^3 Z_2 \\ X_0 & 0 & Y_2 & Z_2 & P & P' \end{pmatrix}$$

$$\begin{pmatrix} 0 & X_2 - \lambda^2 X_0 & Y_0 \\ X_1 & Y_0 & X_2 Y_1 + \lambda X_0 Y_2 \end{pmatrix}$$

and the polynomials

$$Y_0 Y_1 - X_2 Z_1 - \lambda^2 X_0 Z_2$$
$$Y_0 Z_1 - X_2 Y_1{}^2 - \lambda X_0 Y_1 Y_2$$
$$Y_0 Z_2 - (X_0 + \lambda X_2)(Y_2{}^2 + \alpha_0 X_2{}^2 Y_2 + \alpha_1 X_2{}^4) - \lambda \varepsilon_0 X_2 Y_0 Y_2 - \varepsilon_1 X_2{}^3 Y_0$$
$$Z_1{}^2 - Y_1{}^3 - X_1 S$$
$$Z_2{}^2 - Y_2(Y_2{}^2 + \alpha_0 X_2{}^2 Y_2 + \alpha_1 X_2{}^4) - \varepsilon_0 Y_0 Y_2{}^2 - \varepsilon_1 X_2{}^3 Z_2 - X_0 T$$

where $\lambda \neq 0$ if $n = 2$, $\lambda = 0$ if $n = 3$ and

$$P = \varepsilon_1 X_2 Y_0 + \lambda \alpha_0 X_2 Y_2 + \lambda \alpha_1 X_2{}^3$$

$$P' = \varepsilon_1 X_2^{\,2} Y_1 + \lambda\varepsilon_0 Y_1 Y_2 + \lambda Y_0(\alpha_0 Y_2 + \alpha_1 X_2^{\,2})$$
$$S = \gamma_0 X_1^{\,5} + \gamma_1 X_1^{\,3} Y_1 + \gamma_2 X_1 Y_1^{\,2} + \lambda^3 \beta_0 X_0 X_2^{\,4} + \lambda^3 \beta_1 X_0 X_2^{\,2} Y_2 + \lambda^3 \beta_2 X_0 Y_2^{\,2}$$
$$T = \gamma_0 X_0 Y_1 Y_2 + \gamma_1 X_0 X_1^{\,2} Y_2 + \gamma_2 X_0^{\,2} X_1^{\,3} + \beta_0 X_2^{\,5} + \gamma_3 X_2^{\,3} Y_2 + \gamma_4 Y_2^{\,2} + \gamma_5 X_2^{\,2} Z_2$$

Here the coefficients γ_3 , γ_4, γ_5 are such that

$$\begin{cases} \lambda^2 \gamma_3 = \lambda^2 \beta_1 + \lambda\, \alpha_0 \\ \lambda^2 \gamma_4 = \lambda^2 \beta_2 + \lambda\, \alpha_1 \\ \lambda^2 \gamma_5 = \lambda\, \varepsilon_1 \end{cases}$$

Proof This comes from combining (3.2), (3.3) and corollary (3.4).

Remarks (1) Case (i) with $\gamma \neq 0$ and case (ii) with $\lambda \neq 0$ are the same.

(2) Although the coordinate change in corollary (3.4) seems quite arbitrary it is not. If one makes the calculations at the same time for case (ii), trying to make the ring $T = R(F-A_1-A_2, K_F)$ look similar in both cases, this is the kind of result one obtains.

(3) (3.1) has an equivalent statement (3.1)′ that gives a simultaneous presentation for cases (i) and (ii).

(3.1)′ Theorem If F is analytically of type II then $R(F, K_F)$ can be presented as

$$R(F, K_F) = k[X_0, X_1, X_2, Y_0, Y_1, Y_2, Z_1, Z_2] / M$$

where M is the ideal generated by the polynomials in table II-1 (next page) and λ , γ are such that $4\lambda^5 - 2\lambda^3\gamma - \lambda = 0$, and $\lambda \neq 0$ or $\gamma \neq 0$.

Furthermore

(i) if F has a standard numerical decomposition $F = A_1 + A_2 + D$ with A_1 and A_2 disjoint, $\gamma \neq 0$ and $\lambda \neq 0$;

(ii) if F has a standard numerical decomposition $F = A_1 + A_2 + D$ with $A_1 \supset A_2$, $\gamma = 0$ and $\lambda \neq 0$;

(iii) if F has a standard numerical decomposition $F = A_1 + A_2 + A_3 + D$ then $\lambda = 0$ and thus $\gamma \neq 0$.

Table II-1

$-\lambda X_0X_2 + 2\gamma X_0X_1$

$X_0^2 + \lambda X_0X_2$

$X_1X_2 + (4\lambda^3 - 2\lambda\gamma) X_0X_1$

X_1Y_0

X_0Y_0

$(1-2\lambda^2\gamma)X_0Y_1 - 2\lambda^2 X_0Y_2$

$X_1(Y_2 - 2\lambda^2 Y_1) - X_0Y_1$

$(1-2\lambda^2\gamma)X_0Z_1 - 2\lambda^3 X_0Z_2$

$X_1(Z_2 - 2\lambda Z_1) - 2\lambda\gamma X_0Z_1 - 2\lambda^2 X_0Z_2$

$Y_0^2 - X_2^2 Y_1 - (4\lambda^3 - 2\lambda\gamma) X_0X_2Y_1$

$Y_0Y_1 - X_2Z_1 - (4\lambda^3 - 2\lambda\gamma) X_0Z_1$

$Y_0Y_2 - \lambda X_2Z_2 - X_0Z_2$

$Y_1Y_2 - \lambda^2Y_2^2 - \lambda^2(1-2\lambda^2\gamma)Y_1^2 - \lambda\alpha X_2^2Y_0 - \lambda(X_0+\lambda X_2)P$

$Y_0Z_1 - X_2Y_1^2 - (4\lambda^3 - 2\lambda\gamma) X_0Y_1^2$

$Y_0Z_2 - (X_0 + \lambda X_2)(Y_2^2 + (1-2\lambda^2\gamma)Y_1^2 + X_2P) - \alpha X_2^3Y_0$

Table II-1 (continued)

$$Y_2Z_1 - \lambda Y_1Z_2$$

$$Y_1Z_2 - \lambda^2Y_2Z_2 - \lambda(1-2\lambda^2\gamma)Y_1Z_1 - \lambda Y_0P - \alpha X_2Y_0{}^2$$

$$Z_1{}^2 - Y_1{}^3 - X_1S$$

$$Z_1Z_2 - \lambda Y_1Y_2{}^2 - \lambda(1-2\lambda^2\gamma)Y_1{}^3 - \alpha X_2{}^2Y_0Y_1 - (X_0+\lambda X_2)Y_1P -$$
$$- (4\lambda 3-2\lambda\gamma)\, X_0S + 2\lambda^2\, X_1S$$

$$Z_2{}^2 - Y_2{}^3 - (-4\lambda^4+4\lambda^2\gamma)Y_1{}^2Y_2 - \alpha X_2{}^3Z_2 - X_2Y_2P - 4\lambda^2X_1\, S -$$
$$- 2\lambda(2\lambda^2\gamma)\, X_0\, (Y_2\, P + \alpha\, X_2{}^2Z) - X_0T - X_1X_2M$$

with $P = \alpha_0X_2{}^3 + \alpha_1X_2Y_2 + \lambda\alpha_2Z_2$

$S = \delta_0X_1{}^5+\delta_1X_1{}^3Y_1+\lambda^3\delta_2X_0X_2{}^4+\lambda^3\delta_3X_0X_2{}^2Y_2+\lambda^3\delta_4X_0Y_2{}^2$

$T = \delta_2X_2{}^5 + \delta_3X_2{}^3Y_2 + \delta_4X_2Y_2{}^2$

$M = \lambda(4-8\lambda^2\gamma + 4\gamma^2)\, (\delta_0X_1{}^4 + \delta_1X_1{}^2Y_1)$

and $4\lambda^5 - 2\lambda^3\gamma - \lambda = 0$.

Proof of (3.1)′

The statement will follow from combining (3.2) and (3.5) below.

Preliminary considerations for (3.2), (3.3)

Let F be analytically of type II. By (1.14) a standard decomposition of F, $F = \sum_{i=1}^{n} A_i + D$ is such that either $n = 2$ and $\mathcal{O}_{E_2}(A_1) \cong \mathcal{O}_{E_2}$ or $n = 3$ and the decomposition can be chosen such that $\mathcal{O}_{E_2}(A_1) \cong \mathcal{O}_{E_2}$ if E_2 is the elliptic tail contained in A_2. In this case (because F is of type II) $\mathcal{O}_{E_3}(A_1 + A_2) \not\cong \mathcal{O}_{E_3}$ if E_3 is the elliptic tail contained in A_3.

Recall the description of F given for these two cases in (II.4.7) (II.4.11).

If $n = 2$, A_2 and D have no common components and either $A_1 \supset A_2$ or A_1 and A_2 are disjoint and A_1 and D also have no common components.

If $n = 3$, $F = A_1 + A_2 + A_3 + C$ with $C \cong \mathbb{P}^1$. By (1.14) either $A_1 \supset A_3$ or $A_2 \supset A_3$ and A_1 and A_2 are disjoint. By choosing the decomposition of F, we will assume that $A_1 \supset A_3$, A_2 is disjoint from A_1.

By theorem (1.18) $R = R(F, K_F)$ is isomorphic to $k[X_0, X_1, X_2, Y_0, Y_1, Y_2, Z_1, Z_2] / L$, but this will also come from the calculation that follows.

Let $S = R(F-A_1, K_F)$ and $T = \operatorname{Im} f$ where $f : S \to R(F-(A_1+A_2), K_F)$ is the map induced by the restriction maps f_n. By propositions (1.2) and (1.9) we can apply the technique explained in (II.1) to recover S from T and R from S.

Now $\operatorname{Im} f_1$ has codimension 1 and f_n is surjective for $n > 1$. The ring $T = \oplus \operatorname{Im} f_n$ will then be a subring of the ring $B = R(F-(A_1+A_2),K_F)$, coinciding with the ring B in degrees > 1.

We are going to describe the ring R separately for $F-(A_1+A_2)$ 2-connected and $F-(A_1+A_2) = A_3 + C$. It is possible to do the two cases together but the calculations become very complicated. In both cases we will obtain T by elimination from B. Once we have the ring T the calculations are quite mechanical as in the calculations of type III rings.

(3.2) Proposition If F is analytically of type II and a standard numerical decomposition has $n = 3$ the ring $R(F, K_F)$ can be presented as

$$R(F, K_F) = k[X_0, X_1, X_2, Y_0, Y_1, Y_2, Z_1, Z_2] / L$$

where L is generated by the $2{\times}2$ minors of the matrices

$$\begin{pmatrix} 0 & X_0 & X_1 & Y_0 & Y_1 & Z_1 \\ X_0 & Y_2 & 0 & Z_2 & \alpha X_2 Y_0 & \alpha X_2^2 Y_1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & X_2 & Y_0 \\ X_1 & Y_0 & X_2 Y_1 \end{pmatrix}$$

and the polynomials

$$Y_0 Y_1 - X_2 Z_1$$

$$Y_0 Z_1 - X_2 Y_1^2$$

$$Y_0 Z_2 - X_0(Y_2^2 + \alpha_0 X_2^2 Y_2 + \alpha_1 X_2^4) + \alpha X_2^3 Y_0$$

$$Z_1^2 - Y_1^3 - X_1^2(\beta_0 X_1^2 Y_1 + \beta_1 X_1^4)$$

$$Z_2(Z_2 + \alpha X_2^3) - Y_2^3 - X_2^2(\alpha_0 Y_2^2 + \alpha_1 X_2^2 Y_2) - X_0 X_2(\gamma_0 X_2^4 + \gamma_1 X_2^2 Y_2 + \gamma_2 Y_2^2)$$

(Table II-2 in next page lists explicitly a set of generators for L.)

Table II-2

$$X_0^2$$

$$X_0X_1$$

$$X_1X_2$$

$$X_0Y_0$$

$$X_0Y_1$$

$$X_1Y_0$$

$$X_1Y_2$$

$$X_0Z_1$$

$$X_1Z_2$$

$$Y_0^2 - X_2^2Y_1$$

$$Y_0Y_1 - X_2Z_1$$

$$Y_0Y_2 - X_0Z_2$$

$$Y_1Y_2$$

$$Y_0Z_1 - X_2Y_1^2$$

$$Y_0Z_2 - X_0(Y_2^2 + \alpha_0X_2^2Y_2 + \alpha_1X_2^4) + \alpha\, X_2^3\, Y_0$$

$$Y_1Z_2 + \alpha\, X_2Y_0^2$$

$$Y_2Z_1$$

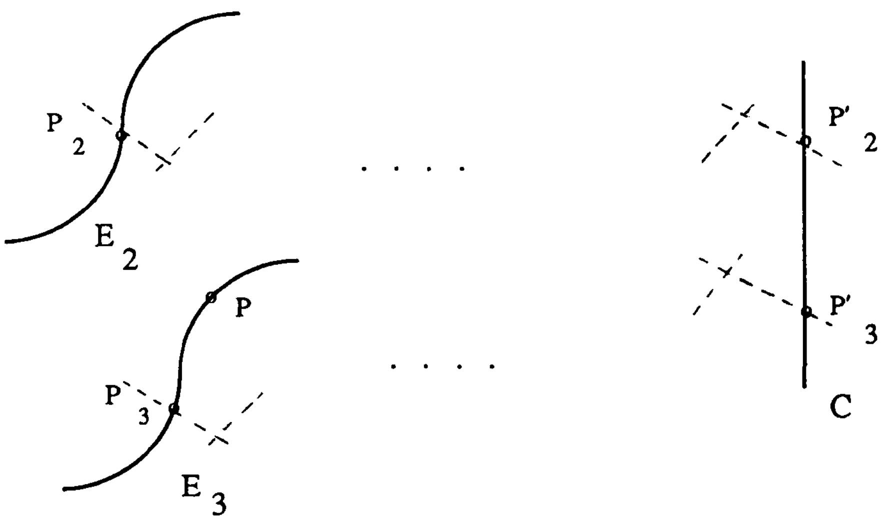
$$Z_1^2 - Y_1^3 - X_1^2(\beta_0X_1^2Y_1 + \beta_1X_1^4)$$

$$Z_1Z_2 + \alpha\, X_2^2Y_0Y_1$$

$$Z_2(Z_2+\alpha X_2^3)-Y_2^3-X_2^2(\alpha_0Y_2^2+\alpha_1X_2^2Y_2)-X_0X_2(\gamma_0X_2^4+\gamma_1X_2^2Y_2+\gamma_2Y_2^2)$$

Proof of (3.2)

A_1 and A_2 are disjoint and we have assumed that A_2 and A_3 are also disjoint. Then by (3.12) A_2 and A_3 consist of elliptic tails E_2 and E_3 linked to C by a simple chain of (-2)-curves (possibly 0).



Let $P_2 = E_2 \cap (A_2+C-E_2)$, $P'_2 = C \cap A_2$, $P_3 = E_3 \cap (A_3+C-E_3)$, $P'_3 = C \cap A_3$. Since $\mathcal{O}_{E_3}(A_1+A_2) \not\cong \mathcal{O}_{E_3}$, $\mathcal{O}_{E_3}(K_F) = \mathcal{O}_{E_3}(P)$ with $P \neq P_3$ and because $\mathcal{O}_{E_2}(A_1+A_2) \cong \mathcal{O}_{E_2}$, $\mathcal{O}_{E_2}(K_F) = \mathcal{O}_{E_2}(P_2)$. Thus by continuity any section of K_F vanishes on P'_2, and P'_3 is not a common zero of the global sections of K_F.

Step 1 of the proof of (3.2) - The ring T

Let $B = \oplus B_n$ and $r_n : B_n = H^0(A_3+C, nK_F) \to H^0(C, nK_F) = H^0(\mathbb{P}^1, \mathcal{O}(1))$ be the restriction maps. $\operatorname{Ker} r_n = H^0(A_3, nK_F-C) \cong H^0(E_3, \mathcal{O}_{E_3}((A_1+A_2)+(n-1)K_F))$ and so $\dim \operatorname{Ker} r_n = n-1$, for every $n \geq 1$ and r_n is surjective for every n. The restriction maps from B_n to $H^0(E_3, nK_F)$ are also surjective.

Let then s_0, s_1 be a basis for B_1 such that s_0 generates $\operatorname{Im}\Gamma_1$. By continuity $s_0(P'_2) = 0$, $s_0(P_3) \neq 0$, $s_0(P'_3) \neq 0$, $s_0(P) = 0$, and $s_{0|E_3}$ generates $H^0(E_3,K_F)$. Also $s_{1|E_3}$ will depend on $s_{0|E_3}$ and we can choose s_1 such that $s_{1|E_3} = 0$. Then $s_1(P'_3) = 0$ and thus necessarily $s_1(P'_2) \neq 0$. Since $K_{F|C}$ is just $\mathcal{O}_{\mathbb{P}^1}(1)$ there does not exist any relation in B involving only expression on s_0, s_1.

By (I.7.3) $R(E,K_F) = \oplus H^0(E_3,K_F) \simeq k[X,Y,Z] / (F_6)$ with $\deg X = 1$, $\deg Y = 2$, $\deg Z = 3$. Since the restriction map $B \to R(E, K_F)$ is surjective, B will have at least two new generators t and u in degrees 2 and 3 (resp.), such that s_0, t, u restricted to $R(E, K_F)$ generate $R(E, K_F)$. These generators t and u can be chosen so that $t \in \operatorname{Ker}\{r_2 : B_2 \to H^0(C,2K_F)\}$ and $u \in \operatorname{Ker}\{r_3 : B_3 \to H^0(C,3K_F)\}$. Then t, u restricted to E_3 will both vanish on P_3 and in B we will have $s_1 t = 0$, $s_1 u = 0$.

Now by considering both restriction maps from A_3+C to E_3 and C and counting dimensions it is easy to check that B has no more independent generators and that there is just another independent relation between elements of B (coming from the unique relation between elements of $R(E, K_F)$) which will be of the form

$$u^2-\lambda t^3-\alpha'_0 s_0^2 t^2 - a'_1 s_0^4 t - a'_2 s_0^3 u - a'_3 s_0 tu = 0 \quad \text{with } \lambda \neq 0 .$$

Changing u and t appropriately we can assume that this relation is

$$u^2-t^3-\alpha_0 s_0^2 t^2-\alpha_1 s_0^4 t-\alpha_2 s_0^3 u = 0 .$$

Then $B = k[S_0,S_1,T,U] / L$ where $k[S_0,S_1,T,U]$ is the weighted polynomial ring with $\deg S_0 = \deg S_1 = 1$, $\deg T = 2$, $\deg U = 3$ and L is the ideal generated by the $2{\times}2$ minors of the matrix

$$\begin{pmatrix} 0 & T & U \\ S_1 & U-\alpha_2 S_0^3 & T^2+\alpha_0 S_0^2 T+\alpha_1 S_0^4 \end{pmatrix}$$

Thus letting $X_2 = S_0$, $Y_0 = S_0S_1$, $Y_1 = S_1^2$, $Y_2 = T$, $Z_1 = S_1^3$, $Z_2 = U$

we have

$T = k[X_2, Y_0, Y_1, Y_2, Z_1, Z_2]/I$ where I is generated by

$F_1{}^4 : Y_0^2 - X_2^2Y_1$

$F_2{}^4 : Y_0Y_1 - X_2Z_1$

$F_3{}^4 : Y_0Y_2$

$F_4{}^4 : Y_1Y_2$

$F_1{}^5 : Y_0Z_1 - X_2Y_1^2$

$F_2{}^5 : Y_0Z_2$

$F_3{}^5 : Y_1Z_2$

$F_4{}^5 : Y_2Z_1$

$F_1{}^6 : Z_1^2-Y_1^3$

$F_2{}^6 : Z_1Z_2$

$F_3{}^6 : Z_2^2-Y_2^3-\alpha_0X_2^2Y_2^2-\alpha_1X_2^4Y_2-\alpha_2X_2^3Z_2$

It will be important to remember when we recover the rings R and S that by our previous choices $X_{2|E_3}$, $Y_{2|E_3}$, $Z_{2|E_3}$ generate $R(E_3, K_F)$, $Y_{0|E_3} = Y_{1|E_3} = $ $= Z_{1|E_3} = 0$ .Remark also that,since $s_1(P'_2) \neq 0$, neither Y_1 or Z_1 vanish on P'_2 (P'_2 is the point where C meets A_2). We have also seen that $s_0(P'_2) = 0$ and thus Y_0 vanishes on P'_2. By our choices of t, u in B, Y_2, Z_2 will also vanish on P'_2.

The ideal I can be presented also as the ideal generated by the 2×2 minors of the matrices

$$\begin{pmatrix} X_2 & Y_0 & Y_1 & Y_2 & Z_1 & Z_2 \\ Y_0 & X_2Y_1 & Z_1 & 0 & Y_1^2 & 0 \end{pmatrix}$$

and $$\begin{pmatrix} 0 & Y_2 & Z_2 \\ Y_1 & Z_2-\alpha_2X_2^3 & Y_2^2+\alpha_0X_2^2Y_2+\alpha_1X_2^4 \end{pmatrix}.$$

Step 2 of the proof of (3.2) The ring S

Once we have the ring T by (1.9) we can apply (II.1) and thus $S = k[X_1,X_2,Y_0,Y_1,Y_2,Z_1,Z_2] / \tilde{I} = k[x_1,x_2,y_0,y_1,y_2,z_1,z_2]$ where $\deg X_1 = \deg X_2 = 1$, $\deg Y_0 = \deg Y_1 = \deg Y_2 = 2$, $\deg Z_1 = Z_2 = 3$ and $x_1 \in \mathrm{Ker}(S_1 \to T_1)$.

To be able to describe $\tilde{I}$ we will need to see how the generators of S restrict to $R(E_2, K_F)$. Since E_2 is not contained in $A_3 + C$, $x_{1|E_2}$ generates $H^0(E_2, K_F)$ and thus $x_{2|E_2} = ax_{1|E_2}$, for some $a \in K$.

Since $\mathcal{O}_{E_2}(K_F) = \mathcal{O}_{E_2}(P_2)$, by continuity we have that $x_{1|E_2}$, $y_{1|E_2}$, $z_{1|E_2}$ generate $R(E_2, K_F)$ and that $y_{0|E_2} = bx_1^2$, $y_{2|E_2} = cx_1^2$,

$z_{2|E_2} = dx_1^3 + ex_1y_1$ (because by continuity $y_0(P_2) = y_2(P_2) = z_2(P'_2) = 0$) .

Changing elements of S by multiples of x_1 does not affect T or I and thus we will assume that $x_{2|E_2} = y_{0|E_2} = y_{2|E_2} = z_{2|E_2} = 0$. Then $J_S = \{s \in S : x_1 s = 0\}$ (see II.1) is generated by x_0, y_0, y_2, z_2 , $(x_1) = x_1 (k[x_1,y_1] + z_1 k[x_1,y_1])$, and (x_1) is isomorphic as a k-vector space to $X_1(k[X_1,Y_1] + Z_1k[X_1,Y_1])$. Thus $\tilde{I}$ will be generated by

$$X_1X_2$$
$$X_1Y_0$$
$$X_1Y_2$$
$$X_1Z_2$$

and $\tilde{F}_i^j = F_i^j - X_1 P_i^j$ where the F_i^j are as in step 1 and P_i^j is a polynomial of deg j-1 in X_1, Y_1, Z_1 . With the choices made so far $F_{i|E_2}^j \equiv 0$ for all F_i^j except F_1^6 . Hence $x_1 P_i^j \in J$ and thus $P_i^j \equiv 0$ for all $(i,j) \neq (1,6)$ and $\tilde{F}_1^6$ will be $Z_1^2-Y_1^3-X_1 P_1^6$. $\tilde{F}_1^6$ restricted to E_2 is the deg 6 equation giving E_2 . By a convenient coordinate change of Z_1 and Y_1 by multiples of X_1 we can assume that $P_1^6 = \beta_0X_1^2Y_1^2 + \beta_1X_1^6$ and thus $\tilde{I}$ is generated by the polynomials in Table II-3 (we will denote $\tilde{F}_i^j$ by F_i^j to avoid heavy notation).

Table II-3

$F_1{}^2 : X_1X_2$

$F_1{}^3 : X_1Y_0$

$F_2{}^3 : X_1Y_2$

$F_1{}^4 : Y_0^2 - X_2^2Y_1$

$F_2{}^4 : Y_0Y_1 - X_2Z_1$

$F_3{}^4 : Y_0Y_2$

$F_4{}^4 : Y_1Y_2$

$F_5{}^4 : X_1Z_2$

$F_1{}^5 : Y_0Z_1 - X_2Y_1^2$

$F_2{}^5 : Y_0Z_2$

$F_3{}^5 : Y_1Z_2$

$F_4{}^5 : Y_2Z_1$

$F_1{}^6 : Z_1^2 - Y_1^3 - X_1^2(\beta_0X_1^2Y_1 + \beta_1X_1^4)$

$F_2{}^6 : Z_1Z_2$

$F_3{}^6 : Z_2^2 - \alpha_2X_2^3Z_2 - Y_2{}^3 - X_2^2(\alpha_0Y_2^2 + \alpha_1X_2^2Y_2)$

Step 3 of the proof of (3.2) The ring R .

Again (by 1.2) we can apply the technique of (II.1) and thus

$$R = k[X_0, X_1, X_2, Y_0, Y_1, Y_2, Z_1, Z_2] / L$$

where $x_0 \in R_1$ is such that $x_{0|F-E_1} \equiv 0$. Since $E_1 = E_3$ and by our previous choices $x_{1|E_3} = y_{0|E_3} = y_{1|E_3} = z_{1|E_3} = 0$, $R(E_3, K_F)$ is generated by $x_{2|E_3}$, $y_{2|E_3}$, $z_{2|E_3}$. Since $x_{0|E_3} = 0$ we will then have

$$J_R = (x_0, x_1, y_0, y_1, z_1) \text{ and } (x_0) = x_0(k[x_2,y_2] + z_2k[x_2,y_2]) ,$$

with (x_0) isomorphic as a k-vector space to $X_0(k[X_2,Y_2] + Z_2\, k[X_2,Y_2])$.

Thus L is generated by

$R_1 : X_0^2$

$R_2 : X_0X_1$

$R_3 : X_0Y_0$

$R_4 : X_0Y_1$

$R_5 : X_0Z_1$

and $\tilde{F}_i^j$ where $\tilde{F}_i^j = F_i^j - X_0\, P_i^j$ where P_i^j is a polynomial of deg j-1 in X_2, Y_2, Z_2.

Because of the nilpotent structure in R it is not possible to determine the P_i^j by looking at the restriction of F_i^j to E_3 , and we will have to use the syzygies between the F_i^j and the fact that $X_0(k[X_2,Y_2] + Z_2k[X_2,Y_2]) \cap L = 0$.

From the syzygy between elements of $\tilde{I}$, $X_2F_2^{\ 3} - Y_2F_1^{\ 2}$ we get $-X_0X_2P_2^{\ 3} + X_0Y_2P_1^{\ 2} \in L$ and thus since $-X_0X_2P_2^{\ 3} + X_0Y_2P_1^{\ 2} \in X_0(K[X_2,Y_2] + Z_2\, K[X_2,Y_2])$ necessarily $X_2P_2^{\ 3} = Y_2P_1^{\ 2}$.

From the syzygy $Y_0F_1{}^2 - X_2F_1{}^3$ we get $-X_0Y_0P_1{}^2 + X_0X_2P_1{}^3 \in L$. Since $X_0Y_0 \in L$ we then have $X_0X_2P_1{}^3 \in L$ and thus $P_1{}^3 \equiv 0$.

Repeating this automatic procedure we have the following:

Syzygy in $\tilde{I}$	**Identities between P_i^j**
$X_2 F_2{}^3 - Y_2 F_1{}^2$	$X_2 P_2{}^3 = Y_2 P_1{}^2$
$X_2 F_1{}^3 - Y_0 F_1{}^2$	$P_1{}^3 = 0$
$Z_2 F_2{}^3 - Y_2 F_5{}^4$	$Z_2 P_2{}^3 = Y_2 P_5{}^4$
$Y_2 F_1{}^4 - Y_o F_3{}^4 + X_2{}^2 F_4{}^4$	$Y_2 P_1{}^4 = -X_2^2 P_4{}^4$
$Y_2 F_2{}^4 - Y_0 F_4{}^4 + X_2 F_4{}^5$	$Y_2 P_2{}^4 = -X_2 P_4{}^5$
$Z_2 F_1{}^4 + X_2^2 F_3{}^5 - Y_0 F_2{}^5$	$X_0(Z_2 P_1{}^4 + X_2^2 P_3{}^5) \in L$
$Z_2 F_3{}^4 - Y_2 F_2{}^5$	$X_0(Z_2 P_3{}^4 - Y_2 P_2{}^5) \in L$
$Z_2 F_4{}^4 - Y_2 F_3{}^5$	$X_0(Z_2 P_4{}^4 - Y_2 P_3{}^5) \in L$
$Y_2 F_1{}^5 - Z_1 F_3{}^4 + X_2Y_1 F_4{}^4$	$P_1{}^5 = 0$
$Y_2 F_2{}^5 - Z_2 F_3{}^4$	$X_0Y_2 P_2{}^5 - X_0Z_2 P_3{}^4 \in L$
$Y_2 F_1{}^6 - Z_1 F_4{}^5 + Y_1^2 F_4{}^4 + Q F_2{}^3$	$P_1{}^6 = 0$
$Y_2 F_2{}^6 - Z_2 F_4{}^5$	$X_0(Y_2 P_2{}^6 - Z_2 P_4{}^5) \in L$
$Y_1 F_3{}^6 - (Z_2 - \alpha X_2^3)F_3{}^5 + (Y_2^2 + P)F_4{}^4$	$X_0(Y_2^2 + P)P_4{}^4 - X_0(Z_2 - \alpha X_2^3)P_3{}^5 \in L$

where

$$Q = \beta_0 X_1 Y_1^2 + \beta_1 X_1^5$$

$$P = \alpha_0 X_2^2 Y_2 + \alpha_1 X_2^4 .$$

Analyzing these identities and using $X_0 \tilde{F}_3{}^6$ we get

$P_1{}^2 = \varepsilon_0 X_2$

$P_1{}^3 = 0$

$P_5{}^4 = \varepsilon_0 Z_2$

$P_2{}^3 = \varepsilon_0 Y_2$

$P_1{}^4 = \alpha X_2^3$

$P_4{}^4 = -\alpha X_2 Y_2$

$P_2{}^4 = \varepsilon_1 X_2 Y_2 + \varepsilon_2 X_2^3$

$P_1{}^5 = 0$

$P_3{}^5 = -\alpha X_2^2 Z_2$

$P_4{}^5 = -\varepsilon_1 Y_2^2 - \varepsilon_2 X_2^2 Y_2$

$P_2{}^6 = -\varepsilon_1 Y_2 Z_2 - \varepsilon_2 X_2^2 Z_2$

$P_1{}^6 = 0$

$P_2{}^5 = \gamma\alpha_1 X_2^4 + \gamma\alpha_0 X_2^2 Y_2 + \gamma Y_2^2 + c X_2 Z_2$

$P_3{}^4 = -\gamma\alpha_2 X_2^3 + \gamma Z_2 + c X_2 Y_2$

where $\alpha_0, \alpha_1, \alpha_2$ are the coefficients of $F_3{}^6$ and

$$P_3{}^6 = \gamma_0 X_2^5 + \gamma_1 X_2^3 Y_2 + \gamma_2 X_2 Y_2^2 + \gamma_3 Y_2 Z_2 + \gamma_4 X_2^2 Z_2 .$$

Making coordinate changes by multiples of X_0

$(X_1 \to X_1 - \varepsilon_0 X_0, Y_1 \to Y_1 + \alpha X_0 X_2, Z_1 \to Z_1 + \varepsilon_1 X_0 Y_2 + \varepsilon_2 X_0 X_2^2$

$Y_0 \to Y_0 - c X_0 X_2)$ we can assume that

$$P_1{}^2 = P_1{}^3 = P_2{}^3 = P_1{}^4 = P_2{}^4 = P_4{}^4 = P_1{}^5 = P_3{}^5 = P_4{}^5 = P_2{}^6 = 0$$

and $P_3{}^4 = \gamma\alpha_2 X_2^3 + \gamma Z_2$ $P_2{}^5 = \gamma\alpha_1 X_2^4 + \alpha_0 X_2^2 Y_2 + \gamma Y_2^2$. We can also change Z_2 by multiples of X_0 so that $P_3{}^6$ is $\gamma_0 X_2^5 + \gamma_1 X_2^3 Y_2 + \gamma_2 X_2 Y_2^2$.

Let us remark that γ must be different from zero. Otherwise y_0, x_0x_1, $y_1 \in \mathrm{Ker}\{H^0(F,2K_F) \to H^0(2E_3,2K_F)\}$ which is 2-dimensional. Thus we can assume that $\gamma = 1$ and changing Z_2 to $Z_2 - \alpha X_2^3$ we have L given by table II-2

End of proof of (3.2)

(3.3) **Proposition** If F is analytically of type II and a standard decomposition of F has n = 2 , the ring $R(F, K_F)$ can be presented as $R(F, K_F) = k[X_0, X_1, X_2, Y_0, Y_1, Y_2, Z_1, Z_2] / L$ where L is the ideal generated by the polynomials in table II-4 (next page) and where $\gamma \neq 0$ if F contains two distinct elliptic tails and $\gamma = 0$ if F contains one elliptic tail appearing with multiplicity 2.

Remark These polynomials with some changes can be presented as the 2×2 minors of some matrices (see statement of (3.1)).

Table II-4

$X_0^2+2\gamma X_0X_1$	$Y_0^2-X_2^2Y_1$
X_0X_2	$Y_0Y_1-X_2Z_1$
X_1X_2	$Y_0Y_2-X_2Z_2$
X_0Y_0	$Y_0Z_1-X_2\,Y_1^2$
X_1Y_0	$Y_0Z_2-X_2Y_1Y_2$
$X_0(Y_2+\gamma Y_1)$	$Y_2Z_1-Y_1Z_2$
$X_1(Y_2-\gamma Y_1)-X_0Y_1$	$Y_2Z_2-\gamma^2Y_1Z_1-Y_0P$
$X_0(Z_2+\gamma Z_1)$	$Z_1^2-Y_1^3-X_1(X_1Q_1+X_0Q_2)$
$X_1(Z_2-\gamma Z_1)-X_0Z_1$	$Z_1Z_2-Y_1^2Y_2-(X_0+\gamma X_1)(X_1Q_1+X_0Q_2)$
$Y_2^2-\gamma^2Y_1^2-X_2\,P$	$Z_2^2-Y_1Y_2^2-\gamma^2X_1(X_1Q_1+X_0Q_2)$

where $\gamma\neq 0$ if F contains 2 distinct elliptic tails, $\gamma = 0$ otherwise,

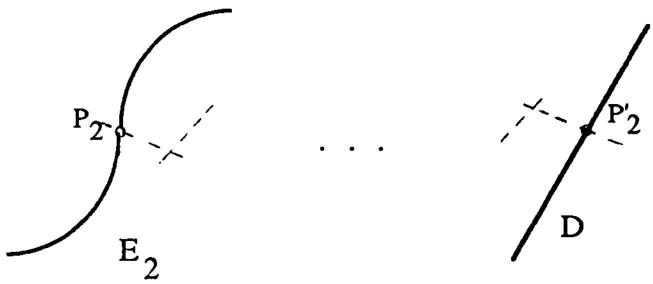
and

$$P=(\alpha_0X_2^3+\alpha_1X_2Y_0+\alpha_2X_2Y_1+\alpha_3Z_1)$$

$$Q_1=\beta_0Y_1^2+\beta_1X_1^2Y_1+\varepsilon^2X_1^4 \quad (\varepsilon\neq 0)$$

$$Q_2=\gamma_0Y_1^2+\gamma_1X_1^2Y_1+\gamma_2X_1^4$$

Proof In this case A_2 has no common components with D and if E_2 is the elliptic tail contained in A_2, E_2 is linked to D by a simple chain of (-2)-curves (possibly 0).



If $P_2 = E_2 \cap (A_2+D-E_2)$, $P'_2 = A_2 \cap D$ we have $\mathcal{O}_{E_2}(K_F) \cong \mathcal{O}_{E_2}(P_2)$, and thus every section of K_F vanishes on P'_2.

If A_1 is disjoint from A_2 we have the same picture with $P'_1 \neq P'_2$ and so also every section of K_F vanishes on P'_1.

Step 1 of the proof of (3.3) The ring T .

D is a 2-connected genus 1 divisor and K_F is a sheaf of deg 2 on D. Then by (I.7.3) $B = R(D,K_F) = k[S_0,S_1,T] / (F_4)$ where $k[S_0,S_1,T]$ is the weighted polynomial ring with deg S_0 = deg S_1 = 1 , deg T = 2 and F_4 can be chosen (by appropriate choice of T) to be

$$T^2-\alpha_0 S_0^4-\alpha_1 S_0^3 S_1-\alpha_2 S_0^2 S_1^2-\alpha_3 S_0 S_1^3-\gamma^2 S_1^4 .$$

If we let $s_0 \in H^0(D,K_F)$ be a generator of $\operatorname{Im} \gamma_1$ we can easily describe $T = \oplus \operatorname{Im} \gamma_n$. Let $X_2 = S_0$, $Y_0 = S_0 S_1$, $Y_1 = S_1^2$, $Y_2 = T$, $Z_1 = S_1^3$, $Z_2 = S_1 T$. Then $T = k[X_2, Y_0, Y_1, Y_2, Z_1, Z_2] / I$

where $\deg X_2 = 1$, $\deg Y_0 = \deg Y_1 = \deg Y_2 = Z_1 \deg Z_1 = \deg Z_2 = 3$ and I is generated by the polynomials in Table II-5 (on next page).

Table II-5

$F_1{}^4 : Y_0^2 - X_2^2 Y_1$

$F_2{}^4 : Y_0 Y_1 - X_2 Z_1$

$F_3{}^4 : Y_0 Y_2 - X_2 Z_2$

$F_4{}^4 : Y_2^2 - \gamma^2 Y_1^2 - X_2 \, (\alpha_0 X_2^3 + \alpha_1 X_2 Y_0 + \alpha_2 X_2 Y_1 + \alpha_3 Z_1)$

$F_1{}^5 : Y_0 Z_1 - X_2 Y_1^2$

$F_2{}^5 : Y_0 Z_2 - X_2 Y_1 Y_2$

$F_3{}^5 : Y_2 Z_1 - Y_1 Z_2$

$F_4{}^5 : Y_2 Z_2 - \gamma^2 Y_1 Z_1 - Y_0 (\alpha_0 X_2^3 + \alpha_1 X_2 Y_0 + \alpha_2 X_2 Y_1 + \alpha_3 Z_1)$

$F_1{}^6 : Z_1^2 - Y_1^3$

$F_2{}^6 : Z_1 Z_2 - Y_1^2 Y_2$

$F_3{}^6 : Z_2^2 - Y_1 Y_2^2$

Remark that the above polynomials (except $F_4{}^4, F_4{}^5$) appear as the 2x2 minors of the matrix

$$\begin{pmatrix} X_2 & Y_0 & Y_1 & Y_2 & Z_1 & Z_2 \\ Y_0 & X_2 Y_1 & Z_1 & Z_2 & Y_1^2 & Y_1 Y_2 \end{pmatrix}$$

Observe that since s_0 and $s_1 \in H^0(D,K_F)$ have no common zeros in D and s_0 generates $\operatorname{Im}\gamma_1$, necessarily y_1 and z_1 do not vanish at the point P_2 where A_2 meets D and if A_1 is disjoint from A_2, y_1 and z_1 will not vanish at the point P_1 where A_1 meets D.

Step 2 of the proof of (3.3) The ring S.

As in step 2 of (3.2) we have $S = k[X_1, X_2, Y_0, Y_1, Y_2, Z_1, Z_2]/\tilde{I}$ where the degrees are the same, $x_{1|E_2}$ generates $H^0(E_2,K_F)$ and by the remark above $R(E_2,K_F)$ is generated by $x_{1|E_2}, y_{1|E_2}, z_{1|E_2}$. Thus $x_{2|E_2} = \varepsilon_0 x_{1|E_2}$, $y_{0|E_2} = (\alpha_1 x_1^2 + \varepsilon_1 y_1)_{|E_2}$, $y_2 = (\alpha_2 x_1^2 + \varepsilon_2 y_2)_{|E_2}$, $z_{2|E_2} = (\alpha_0 x_1^3 + \alpha_1 x_1 y_2 + \varepsilon_3 z_2)_{|E_2}$.

We can change x_2, y_0, y_2, z_2 by a multiple of x_1 and then have $J_S = (x_2, y_0-\varepsilon_1 y_1, y_2-\varepsilon_2 y_1, z_2-\varepsilon_3 z_1)$.

As in step 2 of (3.2) we then have $\tilde{I}$ given by

X_1X_2

$X_1Y_0 - \varepsilon_1 Y_1$

$X_1Y_2 - \varepsilon_2 Y_1$

$X_1Z_2 - \varepsilon_3 Z_1$

and

$\tilde{F}_i^j = F_i^j \; X_1 P_i^j$ with $P_i^j \in \left(k[X_1,Y_1] + Z_1 k[X_1,Y_1]\right)$.

By restricting the $\tilde{F}_i^j$ to $R(E_2,K_F)$ and considering the relations in $R(E_2, K_F)$ obtained in this way we get $\tilde{F}_i^j = F_i^j$ for $j \le 5$ and simultaneously we obtain $y_{0|E_2} = 0$, $y_{2|E_2} = \varepsilon y_{1|E_2}$ with $\varepsilon^2 = \gamma^2$, $z_{2|E_2} = \varepsilon z_{1|E_2}$ and $P_2{}^6 = \varepsilon P_1{}^6$, $P_3{}^6 = \varepsilon^2 P_1{}^6$.

Since $\varepsilon^2 = \gamma^2$, we can assume $\varepsilon = \gamma$ and we then have $\tilde{I}$ generated by the polynomials in table II-6. (We made a coordinate change of Z_1, Z_2, Y_1, Y_2 by multiples of X_1 so that $P_1{}^6$ does not have terms in $X_1Y_1Z_1, X_1^3Z_1$ and the coefficient of $X_1{}^6$ is different from zero).

Table II-6

$F_1{}^2 : X_1 X_2$

$F_1{}^3 : X_1(Y_2-\gamma Y_1)$

$F_2{}^3 : X_1 Y_0$

$F_5{}^4 : X_1(Z_2-\gamma Z_1)$

$F_1{}^4 : Y_0^2 - X_2^2 Y_1$

$F_2{}^4 : Y_0Y_1 - X_2Z_1$

$F_3{}^4 : Y_0Y_2 - X_2Z_2$

$F_4{}^4 : Y_2^2-\gamma^2Y_1^2-X_2(\alpha_0X_2^3+\alpha_1X_2Y_0+\alpha_2X_2Y_1+\alpha_3Z_1)$

$F_1{}^5 : Y_0Z_1 - X_2Y_1^2$

$F_2{}^5 : Y_0Z_2 - X_2Y_1Y_2$

$F_3{}^5 : Y_2Z_1 - Y_1Z_2$

$F_4{}^5 : Y_2Z_2-\gamma^2Y_1Z_1-Y_0(\alpha_0X_2^3+\alpha_1X_2Y_0+\alpha_2X_2Y_1+\alpha_3Z_1)$

$F_1{}^6 : Z_1^2-Y_1^3-X_1^2\,(\beta_0Y_1^2+\beta_1X_1^2Y_1+\varepsilon^2X_1^4)$

$F_2{}^6 : Z_1Z_2 - Y_1^2Y_2 - \gamma X_1^2\,(\beta_0Y_1^2+\beta_1X_1^2Y_1+\varepsilon^2X_1^4)$

$F_3{}^6 : Z_2^2 - Y_1Y_2^2 - \gamma^2X_1^2\,(\beta_0Y_1^2+\beta_1X_1^2Y_1+\varepsilon^2X_1^4)$ with $\varepsilon \neq 0$.

Step 3 of the proof of (3.3) The ring R .

We have two cases to consider: either A_1 and A_2 are disjoint or $A_1 \supset A_2$. We will treat the two cases separately: if A_1 and A_2 are disjoint we will, as in step 2, consider the restriction map to E_1 (E_1 is the elliptic tail contained in A_1) and if $A_1 \supset A_2$ we will, as in step 3 of (3.2), consider the syzygies between the elements of $\tilde{I}$. The two cases could be studied together by considering some of the syzygies, but the calculations get much more complicated.

In both cases, by (1.2), we can again apply the theory of (II.1) and we obtain $R = k[X_0,X_1,X_2,Y_0,Y_1,Y_2,Z_1,Z_2] / L$ where $x_0 \in H^0(F,K_F)$ is such that $x_{0|F-E_1} \equiv 0$.

Case 1 of step 3 $A_1 \supset A_2$.

In this case $E_1 = E_2$, and by step 2 $R(E_2,K_F)$ is generated by $x_{1|E}$, $y_{1|E}$, $z_{1|E}$ and $x_{2|E_2} = y_{0|E_2} = y_2-\gamma y_{1|E_2} = z_2-\gamma z_{1|E_2} = 0$.

So $J = (x_0, x_2, y_0, y_2-\gamma y_1, z_2-\gamma z_1)$ and so L is generated by X_0^2, $X_0X_2, X_0Y_0, X_0(Y_2-\gamma Y_1)$, $X_0(Z_2-\gamma Z_1)$ and $\tilde{F}_i^j = F_i^j - X_0 P_i^j$ where P_i^j is a polynomial of $\deg j-1 \in k[X_1,Y_1]+Z_1\, k[X_1,Y_1]$. ($F_i^j$ as in Table II-5).

Changing X_2, Y_2, Y_0 and Z_2 conveniently by multiples of X_0 we can assume that

$$\begin{aligned}
\tilde{F}_1^{\,2} &= X_1 X_2\\
\tilde{F}_1^{\,3} &= X_1(Y_2-\gamma Y_1) - \varepsilon_1 X_0 Y_1\\
\tilde{F}_2^{\,3} &= X_1 Y_0 - \varepsilon_2 X_0 Y_1\\
\tilde{F}_5^{\,4} &= X_1(Z_2-\gamma Z_1) - \varepsilon_3 X_0 Z_1 .
\end{aligned}$$

Then as in step 3 of (3.2) we will use some of the syzygies between the elements of $\tilde{I}$. Since the procedure is the same we only list the syzygies and the identities obtained.

	Syzygies	**Identities between P_j^i**
(1)	$X_1F_1^4+X_2Y_1F_1^2-Y_0F_2^3$	$P_1^4\equiv 0$
(2)	$X_1F_2^4+Z_1F_1^2-Y_1F_2^3$	$Y_1P_2^3-X_1P_2^4\equiv 0$
(3)	$X_1F_3^4+Z_2F_1^2-Y_2F_2^3$	$\gamma Y_1P_2^3-X_1P_3^4=0$
(4)	$X_1F_4^4-(Y_2+\gamma Y_1)F_1^3+PF_1^2$	$-X_1P_4^4+2\gamma Y_1P_1^3=0$
(5)	$X_1F_1^5-Z_1F_2^3+Y_1^2F_1^2$	$X_1P_1^5+Z_1P_2^3\equiv 0$
(6)	$X_1F_2^5-Z_2F_2^3+Y_1Y_2F_1^2$	$X_1P_2^5+\gamma Z_1P_2^3\equiv 0$
(7)	$X_1F_3^5-Z_1F_1^3+Y_1F_5^4$	$X_1P_3^5-Z_1P_1^3+Y_1P_5^4\equiv 0$
(8)	$X_1F_4^5-(Y_2-\gamma Y_1)F_5^4+PF_2^3$	$X_1P_4^5-PP_2^3\equiv 0$
(9)	$X_1F_2^6-\gamma X_1F_1^6-Z_1F_5^4+Y_1^2F_1^3$	$X_0(X_1P_2^6-\gamma X_1P_1^6-Z_1P_5^4+Y_1^2P_1^3)\in L$
(10)	$X_1F_3^6-\gamma X_1F_2^6-Z_2F_5^4+Y_1Y_2F_1^3$	$X_0(X_1P_3^6-\gamma X_1P_2^6-\gamma Z_1P_5^4+\gamma Y_1^2P_1^3)\in L$

Since P_2^3 is $\varepsilon_2 Y_1$ from $Y_1P_2^3-X_1P_2^4\equiv 0$ we have $\varepsilon_2=0$ and $P_2^4\equiv 0$. Similarly we can obtain the results below

Syzygy	Identities
(1)	$P_1{}^4 = 0$
(2)	$P_2{}^4 = 0$, $P_2{}^3 = 0$
(3)	$P_3{}^4 = 0$
(4)	$P_4{}^4 = 0$ and $\gamma\varepsilon_1 = 0$
(5)	$P_1{}^5 = 0$
(6)	$P_2{}^5 = 0$
(7)	$P_3{}^5 = 0$ and $\varepsilon_1 = \varepsilon_3$
(8)	$P_4{}^5 = 0$

Since $\gamma\varepsilon_1 = 0$ either $\gamma = 0$ or $\varepsilon_1 = 0$. But if $\varepsilon_1 = 0$ we would have $x_2^2, y_0, y_2 - \gamma y_1 \in \operatorname{Ker}(r : H^0(F, 2K_F) \to H^0(2E_2, 2K_F))$ which is 2-dimensional. So $\gamma = 0$ and $\varepsilon_1 \neq 0$.

From (9) we then have $-\varepsilon_1 X_0 Z_1^2 + \varepsilon_1 X_0 Y_1^3 + X_0 X_1 P_2{}^6 \in L$ and thus

$$-\varepsilon_1 X_0 Z_1^2 + \varepsilon_1 X_0 Y_1^3 + X_0 X_1 P_2{}^6 + \varepsilon_1 X_0 \tilde{F}_1{}^6 =$$

$$= X_0 X_1 P_2{}^6 - \varepsilon_1 X_0 X_1^2 (\beta_0 Y_1^2 + \beta_1 X_1^4) - \varepsilon_1 X_0^2 P_1{}^6 \in L$$

hence $P_2{}^6 = \varepsilon_1 X_1 (\beta_0 Y_1^2 + \beta_1 X_1^4)$. Since $\gamma = 0$ we have $P_3{}^6 \equiv 0$.

We make a further coordinate change of X_1 by a multiple of X_1 so that $\varepsilon_1 = 1$ and of Z_1, by multiples of X_0 so that $P_1{}^6$ is $\gamma_0 X_1^5 + \gamma_1 X_1^3 Y_1 + \gamma_2 X_1 Y_1^2$ and thus we have L as in the statement of (3.3) with $\gamma = 0$.

Case 2 of step 3 A_1 and A_2 are disjoint.

Let E_1 be the elliptic tail contained in A_1. Then $x_{0|E_1} \neq 0$ and thus (see end of step 1) $R(E_1, K_F)$ is generated by $x_{0|E_1}, y_{1|E_1}, z_{1|E_1}$. Since making coordinate changes by multiples of x_0 does not affect anything on the ring S or $\tilde{I}$ we can assume that $x_{1|E_1} = \varepsilon\, x_{0|E_1}$, with $\varepsilon \neq 0$, $x_{2|E_1} = 0$, $y_{0|E_1} = \varepsilon_0\, y_{1|E_1}$, $y_{2|E_1} = \varepsilon_1\, y_{1|E_1}$, and $z_{2|E_1} = \varepsilon_2\, z_{1|E_1}$.

Then $J = (x_0-\lambda x_1, x_2, y_0-\varepsilon_0 y_1, y_2-\varepsilon_1 y_1, z_2-\varepsilon_2 z_1)$ with $\lambda \neq 0$ and $(x_0) = x_0(k[x_1,y_1] + z_1\, k[x_1,y_1]) \cong X_0(k[X_1,Y_1] + Z_1\, k[X_1,Y_1])$.

Then L is generated by

$$X_0^2 - \lambda X_0 X_1$$
$$X_0 X_2$$
$$X_0 Y_0 - \varepsilon_0 X_0 Y_1$$
$$X_0 Y_2 - \varepsilon_1 X_0 Y_1$$
$$X_0 Z_2 - \varepsilon_2 X_0 Z_1$$

and

$$\tilde{F}_i^j = F_i^j - X_0\, P_i^j$$

where P_i^j has deg $j-1$ and belongs to $k[X_1,Y_1] + Z_1\, k[X_1,Y_1]$.

By considering, as in step 2, the restrictions of F_i^j to E_1 we obtain (we skip the details)

$P_1{}^2 = P_2{}^3 = P_1{}^4 = P_2{}^4 = P_3{}^4 = P_4{}^4 = P_1{}^5 = P_2{}^5 = P_3{}^5 = P_4{}^5 = 0$, $\varepsilon_2 = \varepsilon_1$,

$\varepsilon_1^2 = \gamma^2$, $P_1{}^3 = \dfrac{\varepsilon_1-\gamma}{\lambda} X_0 Y_1$, $P_5{}^4 = \dfrac{\varepsilon_1-\gamma}{\lambda} X_0 Z_1$, $P_2{}^6 = \dfrac{\varepsilon_1-\gamma}{\lambda} X_0 X_1 \varphi +$

$\varepsilon_1 X_0 P_1{}^6$, $P_3{}^6 = \varepsilon_1^2 X_0 P_1{}^6$.

We can make a coordinate change of Z_1 (and Z_2 accordingly) so that the other relations remain unchanged and ${P_1}^6$ has no terms in Z.

We have $\varepsilon_1^2 = \gamma^2$. If $\varepsilon_1 = \gamma$ or $\gamma = 0$ then we would have x_2^2, y_0, $y_2 - \gamma y_1 \in \mathrm{Ker}\,(r : H^0(F, 2K_F) \to H^0(E_1+E_2, 2K_F)$ which is 2-dimensional. Thus $\varepsilon_1 = -\gamma$ and $\gamma \neq 0$. We will make a further coordinate change of x_1 so that $\lambda = -2\gamma$ and we have L as in the statement of (3.3).

End of proof of (3.3).

(3.4) Corollary Let F be as in (3.3). If A_1 and A_2 are disjoint $R(F, K_F)$ can also be presented as

$$k[X_0, X_1, X_2, Y_0, Y_1, Y_2, Z_1, Z_2] / N$$

where N is generated by

$X_0X_1 - \lambda^2 {X_0}^2$

${X_0}^2 + \lambda X_0X_2$

$X_1X_2 + \lambda {X_0}^2$

X_0Y_0

X_1Y_0

$X_0Y_1 - \lambda^2 X_0Y_2$

$X_1Y_2 - \lambda^2 X_0Y_2$

$X_0Z_1 - \lambda^3 X_0Z_2$

$X_1Z_2 - \lambda^2 X_0Z_2$

${Y_0}^2 - {X_2}^2Y_1 - \lambda X_0X_2Y_2$

$Y_0Y_1 - X_2Z_1 - \lambda^2 X_0Z_2$

$Y_0Y_2 - X_0Z_2 - \lambda X_2Z_2$

$$Y_1Y_2-\lambda^2Y_2^2-\lambda^2\varepsilon_0Y_0Y_2-\lambda\varepsilon_1X_2^2Y_0-\lambda\alpha_0(X_0+\lambda X_2)X_2Y_2-$$
$$-\lambda\alpha_1(X_0+\lambda X_2)X_2^3$$
$$Y_0Z_1 - X_2Y_1^2 - \lambda X_0Y_1Y_2$$
$$Y_0Z_2 - (X_0+\lambda X_2)(Y_2^2+\alpha_0X_2^2Y_2+\alpha_1X_2^4)-\lambda\varepsilon_0X_2Y_0Y_2-\varepsilon_1X_2^3Y_0$$
$$Y_2Z_1 - \lambda Y_1Z_2$$
$$Y_1Z_2-\lambda^2Y_2Z_2-\lambda^2\varepsilon_0Y_0Z_2-\lambda\alpha_0X_2Y_0Y_2-\lambda\alpha_1X_2^3Y_0-\varepsilon_1X_2Y_0^2$$
$$Z_1^2 - Y_1^3 - X_1(X_1Q_1 + \lambda^3X_0Q'_2)$$
$$Z_1Z_2-\lambda Y_1Y_2^2-\lambda\varepsilon_0Y_0Y_1Y_2-\varepsilon_1X_2^2Y_0Y_1-(X_0+\lambda X_2)(\alpha_0X_2Y_2+\alpha_1X_2^3)Y_1-$$
$$-\lambda X_0^2Q_1 - \lambda^3X_0^2Q'_2$$
$$Z_2^2-Y_2^3-\varepsilon_0Y_0Y_2^2-\varepsilon_1X_2^3Z_2-\alpha_0X_2^2Y_2^2-\alpha_1X_2^4Y_2 - X_0^2Q_3 -$$
$$-X_0X_2Q'_2 - X_0X_2Q_4$$

where $\lambda \neq 0$ and

$$\begin{aligned}
Q_1 &= \gamma_0 Y_1^2 + \gamma_1 X_1^2Y_1 + \gamma_2 X_1^4\\
Q'_2 &= \beta_0 Y_2^2 + \beta_1 X_2^2Y_2 + \beta_2 X_2^4\\
Q_3 &= \gamma_0 Y_1Y_2 + \gamma_1 X_1^2Y_2 + \gamma_2 X_0X_1^3\\
\lambda Q_4 &= \alpha_0 Y_2^2 + \alpha_1 X_2^2Y_2 + \varepsilon_1 X_2 Z.
\end{aligned}$$

Proof Consider the presentation of $R(F, K_F)$ given in (3.3). Now we first make the following coordinate change:

$$Y_2 = Y'_2 + BY_1 + CY_0 + DX_2^2$$

$$Z_2 = \lambda Z'_2 + BY_1 + CX_2Y_1 + DX_2Y_0$$

where $B \neq 0, \lambda \neq 0$ and C, D are such that

$2BC = \alpha_3$, $2BD + C^2 = \alpha_2$. Then after making the coordinate change we simplify the relations obtained and we can write (omitting $'$) $R(F, K_F) =$

$k[X_0, X_1, X_2, Y_0, Y_1, Y_2, Z_1, Z_2] / N$ with N generated by

$$X_0^2 + 2\gamma X_0X_1$$

$$X_0X_2$$

$$X_1X_2$$

$$X_0Y_2 + (B+\gamma)X_0Y_1$$

$$X_1Y_2 + (B-\gamma)X_1Y_1 - X_0Y_1$$

$$X_0Y_0$$

$$X_1Y_0$$

$$\lambda X_0Z_2 + (B+\gamma)X_0Z_1$$

$$\lambda X_1Z_2 + (B-\gamma)X_1Z_1 - X_0Z_1$$

$$Y_0^2 - X_2^2Y_1$$

$$Y_0Y_1 - X_2Z_1$$

$$Y_0Y_2 - \lambda X_2Z_2$$

$$Y_2^2 + 2BY_1Y_2+(B-\gamma)(B+\gamma)Y_1^2+\varepsilon_0Y_0Y_2+\varepsilon'_1 X_2^2Y_0+\alpha_0 X_2^2Y_2+\alpha_1X_2^4$$

$$Y_0Z_1 - X_2Y_1^2$$

$$\lambda Y_0Z_2 - X_2Y_1Y_2$$

$$Y_2Z_1 - \lambda Y_1Z_2$$

$$\lambda Y_2Z_2+2B\lambda Y_1Z_2+(B\gamma)(B+\gamma)Y_1^2+\lambda\varepsilon_0Y_0Z_2+\varepsilon'_1X_2Y_0^2+$$

$$+\alpha_0X_2Y_0Y_2+\alpha_1X_2^3Y_0$$

$$Z_1^2 - Y_1^3 - X_1 S$$

$$\lambda Z_1Z_2 - Y_1^2Y_2 - (X_0 - (B-\gamma)X_1) S$$

$$\lambda^2 Z_2^2 - Y_1Y_2^2 - (B-\gamma)^2 X_1 S + 2B X_0S$$

where $S = X_1 Q_1 + X_0 Q_2$.

Now if we let $B = \gamma$ and $\lambda^2 = -\frac{1}{2\gamma}$ and we make a further coordinate change $X_2 = X'_2 + \frac{1}{\lambda}X_0$ we can, by playing around with the relations, present $R(F, K_F)$ as in the statement. The change in S comes from the fact that with these changes of coordinates it is possible to write $X_0Q_2 = X_0Q'_2$ (in R) where Q'_2 is a polynomial in X_2, Y_2. Also the coefficient ε_1 is $\frac{1}{\lambda}\varepsilon'_1$.

(3.5) Corollary Let F be as in (3.3). $R(F, K_F)$ can be presented as $k[X_0, X_1, X_2, Y_0, Y_1, Y_2, Z_1, Z_2] / N$ where N is generated by the polynomials in Table II-1, and the parameters γ, λ satisfy $4\lambda^4-2\gamma\lambda^2-1 = 0$.

Proof

We make the same coordinate change as in (3.5) with B such that $B-\gamma = \frac{1}{B}$ and λ such that $\lambda^2 = -\frac{1}{2B}$ and changing a bit the polynomials by adding two of them or multiplying by λ^2 we get the result.

End of proof.

Section 4. The canonical ring of a type I fibre

(4.1) Theorem If F is analytically of type I the ring $R(F, K_F)$ can be presented as $k[X_0, X_1, X_2, Y_1, Y_2, Z] / I$ where I is generated by 9 polynomials.

(i) If a standard numerical decomposition of F, $F = \sum_{i=1}^{n} A_i + D$, has $n = 1$ and $A_1 \cap D = \{P\}$, (i.e. the elliptic tail E contained in F appears with multiplicity 1), then I is generated by the $2{\times}2$ minors of the matrix

$$\begin{pmatrix} 0 & X_2 & X_1 & Y_1 \\ X_0 & Y_2 & B & Z \end{pmatrix}$$

and the polynomials

$$U : Y_1(Y_1 - X_1(\alpha_0 X_1 + \alpha_1 X_2) - \alpha_2 X_2^2) - X_1 P - X_2^2 Q$$

$$V : Z(Y_1 - X_1(\alpha_0 X_1 + \alpha_1 X_2) - \alpha_2 X_2^2) - B\,P - X_2 Y_2 Q$$

$$W : Z(Z - B\,(\alpha_0 X_1 + \alpha_1 X_2 Y_2) - \alpha_2 X_2 Y_2) - G\,P - Y_2^2 Q - X_0^2 M$$

where $B = \lambda X_2^2 + \lambda\gamma Y_2$

$G = \lambda X_2 Y_2 + \lambda\gamma Z$

$P = \beta_0 X_2^3 + \beta_1 X_1 X_2^2 + \beta_2 X_1^2 X_2 + \beta_3 X_1^3$

$Q = \varepsilon X_2^2 + Y_2$

$M = \varepsilon_0 X_0^4 + \varepsilon_1 X_0^2 Y_2$.

In the above $\lambda=0$ if and only if the unique component Γ of D to which P belongs is such that $\Gamma^2=-3$, $K\Gamma=0$, and $\lambda\gamma=0$ if and only if $h^0(D, \mathcal{O}_D(K_D-2P))\neq 0$.

(ii) If the elliptic tail E contained in F appears with multiplicity bigger than 1, I is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} 0 & X_0 & X_1 & Y_1 \\ X_0 & A & Y_2 & Z \end{pmatrix}$$

and the polynomials

$$U : Y_1(Y_1 - X_1(\alpha_0 X_1+\alpha_1 X_2) - \alpha_2 X_0 X_2)-\lambda X_1(X_1 P_1+X_2 P_2) - X_0 X_2 Q$$

$$V : Z(Y_1 - X_1(\alpha_0 X_1+\alpha_1 X_2) - \alpha_2 X_0 X_2)-\lambda Y_2(X_1 P_1+X_2 P_2) - X_2 AQ-X_0 H$$

$$W : Z(Z - Y_2(\alpha_0 X_1+\alpha_1 X_2) - \alpha_2 X_2 A)-\lambda Y_2^2 P_1 - X_2 GQ-AH-X_0 X_2 M$$

where $\lambda\neq 0$, $P_1, P_2, Q \in k[X_2, Y_2]_2$, $M, H \in k[X_2, Y_2]_4$

$$A = X_1^2 + \delta X_2^2 + \gamma Y_2 + a_0 X_1 X_2 + a_1 Y_1$$

$$G = X_1 Y_2 + a_0 X_2 Y_2 + a_1 Z.$$

and P_2,Q satisfy

$$\lambda Y_2 P_2 = -(\delta X_2^2 + \gamma Y_2)Q.$$

In the above $\gamma a_0 = \gamma a_1 = a_0 a_1 = 0$ and

$\delta\neq 0$ if E appears with multiplicity 2 in F and F-E is 2-connected (i.e. the standard numerical decomposition of F has n = 1)

$\delta = 0$ and $\gamma \neq 0$ — if E appears with multiplicity 2 in F and F-E is not 2-connected (i.e. the standard numerical decomposition of f has n = 2)

$\delta = 0$, $\gamma = 0$ — if E appears with multiplicity 3 in F (i.e. the standard decomposition of F is $F = A_1 + A_2 + A_3 + C$).

Furthermore for this last case $a_0 a_1 = 0$. Also $a_0 \neq 0$ if $\mathcal{O}_E(A_1 + A_2) \cong \mathcal{O}_E$ and $a_1 \neq 0$ if $\mathcal{O}_E(A_1 + A_2) \not\cong \mathcal{O}_E$.

Proof This follows from (4.4), (4.6) below.

(4.2) **Definition** If F is as in (4.1(i)), F is *hyperelliptic* if $\lambda\gamma=0$ and *non-hyperelliptic* if $\lambda\gamma\neq 0$.

Preliminary considerations for (4.3-4.6).

By (1.18), $R(F, K_F)$ is isomorphic to $k[X_0, X_1, X_2, Y_1, Y_2, Z] / L$. Let $S = R(F-A_1, K_F)$. Then $S = k[X_1, X_2, Y_1, Y_2, Z] / I$ and by (1.2) we can apply the techniques of (II.1) to recover R from S (as in sections 2 and 3).

By (1.7), F contains a unique elliptic tail E and for a standard numerical decomposition of F, $F = \sum_{i=1}^{n} A_i + D$, either

(A) A_1 and $F-A_1$ have no common components and A_1 intersects $F-A_1$ transversally in one point P

or

(B) A_1 and $F-A_1$ have common components.

In case (B), one of the following holds:

(B_1) $n = 1$, $A_1 = E$ and E is not contained in D-E

(B_2) $n = 2$, $A_1 \supset A_2$, D is a 2-connected genus 1 divisor without common components with A_2, and $\mathcal{O}_E(A_1) \not\cong \mathcal{O}_E$

(B_3) $n = 3$, $A_1 \supset A_2 \supset A_3$, $D \cong \mathbb{P}^1$ and $\mathrm{Im}\{S_1 \to H^0(D, K_F)\}$ has dim 1 $(\mathcal{O}_E(A_1) \not\cong \mathcal{O}_E$, $\mathcal{O}_E(A_1 + A_2) \cong \mathcal{O}_E)$

or

(B_4) $n = 3$, $A_1 \supset A_2 \supset A_3$, $D \cong \mathbb{P}^1$ and $S_1 \to H^0(D, K_F)$ is surjective $(\mathcal{O}_E(A_1) \not\cong \mathcal{O}_E$ and $\mathcal{O}_E(A_1 + A_2) \not\cong \mathcal{O}_E)$.

We are going to study S and R separately in cases (A) and (B).

In (4.3) and (4.5) we describe S and (4.4) and (4.6) we describe R (in cases (A) and (B) respectively).

(4.3) Proposition If F is analytically of type I and a standard numerical decomposition of F, $F = \sum_{i=1}^{n} A_i + D$, has $n = 1$ and $A_1 \cap D = \{P\}$ then the ring $S = R(D, K_{F|D})$ can be presented as $k[X_1, X_2, Y_1, Y_2, Z] / I$ where I is the ideal generated by the following polynomials:

$$T_0 : X_1Y_2 - \lambda X_2^3 - \lambda\gamma X_2Y_1$$

$$T_1 : Y_1Y_2 - X_2Z$$

$$T_2 : X_1Z - \lambda\gamma Y_1^2 - \lambda X_2^2Y_1$$

$$U : Y_1^2 - X_2^2Y_2 - \gamma_0X_2^4 - \beta_0X_2^2Y_1 - X_1(P_3 + Y_1P_1)$$

$$V : Y_1Z - X_2Y_2^2 - \gamma_0 X_2^3Y_2 - \beta_0X_2^2Z - (\lambda X_2^2+\lambda\gamma Y_1)P_3 - X_1Z P_1$$

$$W : Z^2-Y_2^3-\gamma_0X_2^2Y_2^2-\beta_0X_2Y_2Z-(\lambda X_2Y_2+\lambda\gamma Z) P_3-(\lambda X_2^2+\lambda\gamma Y_1)Z P_1;$$

Here $P_1 = \varepsilon_0 X_1 + \varepsilon_1 X_2$

$$P_3 = \mu_0 X_1^3 + \mu_1 X_1^2 X_2 + \mu_2 X_1 X_2^2 + \mu_3 X_2^3 .$$

Proof If $F = A_1 + D$ with $A_1 \cap D = \{P\}$, then P is a non-singular point of D and $\mathcal{O}_D(K_F) \cong \mathcal{O}_D(K_D + P)$. Since D is 2-connected, K_D is generated by its global sections and $\deg K_{D|\Gamma} \geq 0$, for every component Γ of D. Furthermore $\deg K_{D|\Gamma} = 0$ if and only if $\Gamma \cong \mathbb{P}^1$ and $\Gamma(D-\Gamma) = 2$.

By Serre duality $h^0(D, \mathcal{O}_D(K_D-2P)) = h^1(D, \mathcal{O}_D(2P))$, and by Riemann--Roch $h^0(D, \mathcal{O}_D(2P)) - h^1(D, \mathcal{O}_D(2P)) = 1$.

If D is irreducible, then either $\mathcal{O}_D(K_D-2P) \cong \mathcal{O}_D$ (and P is a Weierstrass point of D) or $h^0(D, \mathcal{O}_D(2P)) = 1$.

If D is not irreducible and $h^0(D, \mathcal{O}_D(K_D-2P)) \neq 0$, by (I.3.1), (I.3.3), then either $\mathcal{O}_D(K_D-2P) \cong \mathcal{O}_D$ or there exists a decomposition of D, $D = A+B$ such that $AB \leq \deg (K_D-2P)_{|B}$.

It is easy to see that $h^0(D, \mathcal{O}_D(K_D-2P)) \neq 0$ and $\mathcal{O}_D(K_D-2P) \not\cong \mathcal{O}_D$ happens if and only if the unique component Γ of D containing P satisfies $\Gamma \cong \mathbb{P}^1$ and $\Gamma(D-\Gamma) = 2$ (i.e $\Gamma^2 = -3$ and $K\Gamma = 1$).

Since K_D is free, by Castelnuovo's lemma, the maps

$$g_m : H^0(D, K_D) \otimes H^0(D, mK_F) \longrightarrow H^0(D, (m+1)K_F-P)$$

and the maps

$$f_m : H^0(D, K_D) \otimes H^0(D, mK_F-P) \longrightarrow H^0(D, (m+1)K_F-2P)$$

are surjective, for $m \geq 2$.

We can consider, for each $m \geq 2$, the following filtration of $H^0(D, mK_F)$:

$$0 \subset H^0(D, mK_F - mP) = H^0(D, mK_D) \subset H^0(D, mK_F-(m-1)P) \subset \dots \subset$$
$$\subset H^0(D, mK_F-P) \subset H^0(D, mK_F)\ .$$

For $m = 1$ we have

$$0 \subset H^0(D, K_F-3P) \subset H^0(D, K_F-2P) \subset H^0(D, K_F-P) = H^0(D, K_F)$$

and either $h^0(D, K_F-3P) = 0$ or $h^0(D, K_F-3P) = 1$. If $h^0(D, K_F-3P) = 1$ then also $h^0(D, K_F-2P) = 1$, because K_D is generated by its global sections.

Let x_1, x_2 be a basis of S_1. We will suppose that in the filtration above, $x_1 \in H^0(D, K_F-2P)$ and $x_2 \in H^0(D, K_F-P) \setminus H^0(D, K_F-2P)$.

In degree 2, x_1^2, x_1x_2, x_2^2 are independent and generate

$H^0(D, 2K_F-2P) = H^0(D, 2K_D) \subset H^0(D, 2K_F)$. The two new generators y_1, y_2 will then belong to $H^0(D, 2K_F)\setminus H^0(D, 2K_F-2P)$, and we will assume that $y_1 \in H^0(D, 2K_F-P)$. Then $y_2 \in H^0(D, 2K_F) \setminus H^0(D, 2K_F-P)$ and we choose y_2 such that y_2 has no common zeros with x_2.

Using the filtration above and the fact that the maps f_m are surjective, we verify that in degree 3

$$(x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1y_1, x_2y_1)$$

form a basis of $H^0(D, 3K_F-2P) \subset H^0(D, 3K_F)$. By the choices made for x_2, y_2, we see that x_2y_2 is a complementary element of $H^0(D, 3K_F-2P)$ in $H^0(D, 3K_F-P)$.

Now x_1y_2 belongs also to $H^0(D, 3K_F-2P)$ and thus we have a relation of the form

$$x_1y_2 = \alpha_1x_1y_1 + \alpha_2x_2y_1 + \alpha_3x_1^3 + \alpha_4x_1^2x_2 + \alpha_5x_1x_2^2 + \alpha_6x_2^3$$

holding in S.

Using the free pencil trick we have that

$$\mathrm{Im}\left\{x_2^2.H^0(D, 2K_F) + y_2.H^0(D, 2K_F) \longrightarrow H^0(D, 4K_F)\right\}$$

has dimension 9 and thus

$$(x_2^4, x_2^3x_1, x_2^3x_1^2, x_1^2y_1, x_1^2y_2, x_1^2y_2, x_1x_2y_1^2, y_1y_2)$$

are linearly independent in $H^0(D, 4K_F)$.

Hence $(x_2^3, x_2^2x_1, x_2x_1^2, x_2y_1, x_2y_2, x_1y_2)$ are independent in $H^0(D, 3K_F)$ and so in the relation above, $\alpha_1 \neq 0$ or $\alpha_3 \neq 0$. Changing y_1 by a multiple of x_1^2 if necessary, we can suppose that $\alpha_3 \neq 0$.

Now if $\alpha_2 = \alpha_6 = 0$, we have a relation of the type $x_1t = 0$ and it is easy to check that this happens if and only if the unique component Γ of D to which P belongs is isomorphic to $\mathbb{P}^1$ and $\Gamma(D-\Gamma) = 2$, giving $h^0(D, K_D-2P) \neq 0$ and $\mathcal{O}_D(K_D-2P) \not\cong \mathcal{O}_D$.

In the other cases we have $\alpha_2 \neq 0$ or $\alpha_6 \neq 0$ and looking at the identifications above it is clear that $\alpha_2 = 0$ if and only if $h^0(D, K_F-3P) = 1$.

Changing y_2 by a multiple of x_2, if necessary, we can suppose that $\alpha_6 = \lambda$, $\alpha_2 = \lambda\gamma$. Then we can write the relation as:

$$x_1y_2 = \delta x_1^3 + \lambda x_2^3 + \lambda\gamma x_2y_1 + a_0x_1y_1 + a_1x_1^2x_2 + a_2x_1x_2^2$$

where $\delta \neq 0$, $\lambda\gamma = 0$ if and only if $h^0(D, K_D-2P) \neq 0$, and $\lambda = 0$ if and only if $h^0(D, K_D-2P) \neq 0$ and $\mathcal{O}_D(K_D-2P) \not\cong \mathcal{O}_D$.

There are no other relations in degree 3 and necessarily the new generator $z \in H^0(D, 3K_F) \setminus H^0(D, 3K_F-P)$.

Since f_3 is surjective, $H^0(D, 4K_F-2P) \subset H^0(D, 4K_F)$ has as a basis

$x_1^4, \ldots, x_2^4, x_1^2y_1, x_1x_2y_1, x_2^2y_1, x_2^2y_2$.

Because $z(P) \neq 0$, $y_2(P) \neq 0$, (x_2z, y_2^2) form a complementary basis for $H^0(D, 4K_F-2P)$ in $H^0(D, 4K_F)$.

Now $y_1^2, x_1z, y_1y_2 \in H^0(D, 4K_F)$ and thus we will have three relations involving these elements.

As $y_1^2, x_1z \in H^0(D, 4K_F-2P)$ we will have

$$y_1^2 = P_4(x_1, x_2) + y_1P_2(x_1, x_2) + \alpha x_2^2y_2$$

$$x_1z = L_4(x_1, x_2) + y_1L_2(x_1, x_2) + \gamma x_2^2y_2 .$$

Since $y_1y_2 \in H^0(D, 4K_F-P) \setminus H^0(D, 4K_F-2P)$ we will have $y_1y_2 = Q_4(x_1, x_2) + y_1 P_2(x_1, x_2) + a_0x_2^2y_2 + \mu x_2z$, with $\mu \neq 0$.

Using again the maps g_m, f_m it is easy to see that

$(x_1^5, \ldots, x_2^5, x_1^3y_1, \ldots, x_2^3y_1, x_2^3y_2, x_2y_2^2, x_2^2z, y_2z)$ form a basis of S_5. Since y_1z also belongs to S_5 we have a relation

$$y_1z = \mu' x_2y_2^2 + Q_5(x_1, x_2) + y_1Q_3(x_1, x_2) + \varepsilon x_2^3y_2 + \varepsilon' x_2^2z ,$$

which is the unique relation in degree 5.

In the same way we have in degree 6 a relation

$$z^2 = \mu'' y_2^3 + Q_6(x_1, x_2) + y_1 Q_4(x_1, x_2) + x_2 P_5(x_2, y_2, z)$$

with $\mu'' \neq 0$, because $z(P) \neq 0$, $y(P) \neq 0$.

By a simple dimension count and looking at the relations established so far, it is easy to see that there are no more relations.

Thus the relations between the elements of S are:

$$T_0 : x_1 y_2 - \delta x_1^3 - \lambda x_2^3 - \lambda\gamma x_2 y_1 - a_0 x_1 y_1 - a_1 x_1^2 x_2 - a_2 x_1 x_2^2 = 0$$

$$U : y_1^2 - p_4 - y_1 p_2 - \alpha\, x_2^2 y_2 = 0$$

$$T_1 : y_1 y_2 - q_4 - y_1 q_2 - \mu_0 x_2^2 y_2 - \mu x_2 z = 0$$

$$T_2 : x_1 z - \ell_4 - y_1 \ell_2 - \beta\, x_2^2 y_2 = 0$$

$$V : y_1 z - \mu' x_2 y_2^2 - q_5 - y_1 q_3 - \varepsilon_0 x_2^3 y_2 - \varepsilon_1 x_2^2 z = 0$$

$$W : z^2 - \varepsilon y_2^3 - q_6 - y_1 q'_4 - b_0 x_2^4 y_2 - b_1 x_2^2 y_2^2 - b_2 x_2 y_2 z - b_3 x_2^3 z = 0$$

where $p_2, q_2, \ell_2 \in k[x_1, x_2]_2$, $q_3 \in k[x_1, x_2]_3$, $p_4, q_4, \ell_4, q'_4 \in k[x_1, x_2]_4$, $q_5 \in k[x_1, x_2]_5$, $q_6 \in k[x_1, x_2]_6$, and $\mu \neq 0$, $\varepsilon \neq 0$.

We are going to find the relations between the coefficients of these polynomials. To make the calculations easier we will first make some coordinate changes.

We change y_2 so that T_0 becomes

$$T : x_1 y_2 - \lambda\, x_2^3 - \lambda\gamma\, x_2 y_1$$

and z so that T_1 becomes

$$T_1 : y_1y_2 - c_0x_1^4 - c_1x_1^2y_1 - x_2z .$$

Remark that for n odd the set

$$\left\{x_1^n ,..., x_2^n, x_1^{n-2}y_1 ,..., x_2^{n-2}y_1 , x_2^{n-2}y_2 ,..., x_2\, y_2^{\frac{n-1}{2}} , x_2^{n-3} z ,..., y_2^{\frac{n-3}{2}} z\right\}$$

forms a basis of S_n and for n even the set

$$\left\{x_1^n ,..., x_2^n , x_1^{n-2}y_1 ,..., x_2^{n-2}y_1 , x_2^{n-2}y_2 ,..., y_2^{\frac{n}{2}} , x_2^{n-3} z ,..., x_2y_2^{\frac{n-4}{2}} z\right\}$$

forms a basis of S_n .

We have in S , $y_1T_0 -x_1T_1 -x_2T_2 + \lambda\gamma\, x_2U = 0$. This gives

$$(1) \qquad (c_0\, x_1^5 + x_2\ell_4-\lambda\gamma\, x_2p_4) + (c_1x_1^3 - \lambda x_2^3 + x_2\ell_2 - \lambda\gamma\, x_2p_2)y_1 +$$

$$+ (\beta-\lambda\gamma\alpha)\, x_2^3y_2 = 0 \text{ in } S .$$

But the elements appearing in (1) are part of a basis of S_5 and thus we get the following identities

$$\begin{cases} c_0 = 0 \\ c_1 = 0 \\ \ell_4 = \lambda\gamma\, p_4 \\ \ell_2 = \lambda\gamma\, p_2 + \lambda\, x_2^2 \\ \beta = \lambda\gamma\alpha \end{cases} .$$

Now write $p_4 = x_1p_3 + \gamma_0x_2^4$ and $p_2 = \beta_0x_2^2 + \beta_1x_1x_2 + \beta_2x_1^2$. In the same way from

$$(y_1-p_2)T_1 - y_2U - (p_3 + \beta\beta_2 x_2^3)T_0 - (\beta_1 x_2^2 + \beta_2 x_1 x_2)T_2 + x_2 V = 0 \quad \text{we get}$$

$$\begin{cases} \beta_0 = \varepsilon_1 \\ \alpha = \mu' \\ \varepsilon_0 = (\gamma_0 + \beta_1\beta) \\ q_5 = (\beta_1 x_2 + \beta_2 x_1)\ell_4 + \lambda\beta\beta_2\, x_2^5 + \lambda\, x_2^2\, p_3 \\ q_3 = (\beta_1\, x_2 + \beta_2\, x_1)\ell_2 + \lambda\gamma\beta\beta_2\, x_2^3 + \lambda\gamma\, p_3 \end{cases}$$

and thus we can write V as

$$V : y_1 z - \alpha\, x_2\, y_2^2 - \beta_0\, x_2 z - \gamma_0\, x_2^3 y - p_1 x_1 z - p_3(\lambda\, x_2^2 + \lambda\gamma\, y_1) = 0$$

where $p_1 = \beta_1\, x_2^2 + \beta_2\, x_1 x_2$.

The same reasoning for the expression obtained from $z\,T_1 - y_2 V + x_2 W$ gives

$$W : z^2 - \alpha y_2^3 - \beta_0 x_2 y_2 z - \gamma_0 x_2^2 y_2^2 - (\lambda x_2^2 + \lambda\gamma y_1) z\, p_1 - (\lambda x_2 y_2 + \lambda\gamma\, z) p_3 = 0.$$

Since $\alpha \neq 0$ we can assume that $\alpha = 1$. Thus S can be presented as stated.

(4.4) Proposition If F is as in (4.3), the ring R can be presented as $k[X_0, X_1, X_2, Y_1, Y_2\, Z] / I$ where I is the ideal generated by the $2{\times}2$ minors of the matrix

$$\begin{pmatrix} 0 & X_2 & X_1 & Y_1 \\ X_0 & Y_2 & \lambda X_2^2+\lambda\gamma Y_1 & Z \end{pmatrix}$$

and the polynomials

$$U : Y_1^2 - X_2^2Y_2 - \gamma_0X_2^4 - \beta_0X_2^2Y_1 - X_1(P_3 + Y_1P_1)$$

$$V : Y_1Z - X_2Y_2^2 - \gamma_0X_2^3Y_2 - \beta_0X_2^2Z - (\lambda X_2^2+\lambda\gamma Y_1)P_3 - X_1Z\,P_1$$

$$W : Z^2 - Y_2^3 - \gamma_0X_2^2Y_2^2 - \beta_0X_2Y_2Z - (\lambda X_2Y_2+\lambda\gamma Z)P_3 -$$
$$- (\lambda X_2^2 + \lambda\gamma Y_1)Z\,P_1 - X_0^4\,Q$$

with $\quad Q = \varepsilon_0X_0^2 + \varepsilon_1Y_2\,,$

$$P_1 = \alpha_0X_1+\alpha_2X_2\,,$$

$$P_3 = \mu_0X_1^3 + \mu_1X_1^2X_2 + \mu_2X_1X_2^2 + \mu_3X_2^3\,.$$

Proof This calculation is similar to the calculations in (2.4(b)) or (2.1(b)) and we omit part of the details.

Applying the techniques of (II.1) to recover R from S we have $R = k[X_0, X_1, X_2, Y_1, Y_2, Z] / \tilde{I}$ where $x_0 \in R$ is such that $x_{0|_{F-E}} = 0$. Since E is not contained in $F-A_1$, $x_{0|_E}$ generates $H^0(E, K_F)$.

Now in S , both Y_2 and Z do not vanish on the point where A_1 meets D . Thus by continuity we have $R(E, K_F)$ generated by $x_{0|_E}$, $y_{2|_E}$, $z_{|_E}$. Since y_1 in S vanishes on P , $y_{1|_E} = \alpha_0\, x_0^2$. So choosing Y_1, X_1, X_2 appropriately we have

$$J_R = \{t \in S : x_0t = 0\} = (x_1, x_2, y_1)\,.$$

Also (x_0) is isomorphic as a k-vector space to $X_0(k[X_0, Y_2] + Z\,k[X_0, Y_2])$.

So $\tilde{I}$ is generated by

$$X_0X_1$$
$$X_0X_2$$
$$X_0Y_1$$

and the lift backs of the generators of I in S . Now with the choices made so far and considering the restriction to E (as in, (2.4.b)), it is easy to see that $\tilde{T}_0 = T_0$, $\tilde{T}_1 = T_1$, $\tilde{T}_2 = T_2$, $\tilde{U} = U$, $\tilde{V} = V$ and $\tilde{W} = W + X_0 P_5$ with $P_5 \in k[X_0, Y_2] + Zk[X_0, Y_2]$. Changing Z and Y_2 by multiplies of X_0 we can assume that

$$P_5 = \alpha_0 X_0^4 Y_2 + \alpha_1 X_0^6$$

(4.5) Proposition If F is analytically of type I and $F\text{-}A_1$ contains an elliptic tail, the ring S can be presented as $k[X_1, X_2, Y_1, Y_2, Z] / I$ where I is the ideal generated by the $2{\times}2$ minors of the matrix

$$\begin{pmatrix} 0 & X_1 & Y_1 \\ A & L & Z \end{pmatrix}$$

and the polynomials

$$U : Y_1(Y_1 - P_3) - X_1(X_1P_1 + X_2P_2 + \beta X_2Y_1)$$

$$V : Y_1(Z - (X_1 + a_0X_2)Q_3) - (L + a_1Q_3)(X_1P_1 + X_2P_2 + \beta X_2Y_1) - X_2Q_2A$$

$$W : Z(Z - X_1Q_3 - X_2Q_4) - P_1 L(L + a_1Q_3) - AH_4 -$$
$$-a_1 X_2Q_5(X_1P_1 + X_2P_2 + \beta X_2Y_1) - X_2(X_1 + a_0X_2)(Q_2L + Y_1Q_5)$$

where $A = X_1^2 + \delta X_2^2 + \gamma Y_2 + a_0 X_1X_2 + a_1 Y_1$

The parameters a_0, a_1, γ, δ are such that one of them is different from zero and satisfy $a_0 a_1 = 0$, $\delta\gamma = \gamma a_0 = \gamma a_1 = 0$.

The polynomials L, P_1, P_2, P_3 Q_2, Q_3, Q_4, Q_5 are polynomials of degree 2 in X_2, Y_2 and H_4 is a polynomial of degree 4 in X_2, Y_2. These polynomials satisfy the following identities (denoting $L+a_1 Q_3$ by Λ) :

$$\Lambda P_2 = -(\delta X_2^2 + \gamma Y_2) Q_2$$

$$\Lambda P_3 = -(\delta X_2^2 + \gamma Y_2) Q_3$$

$$P_2 Q_3 = P_3 Q_2$$

$$Q_2 Q_4 = Q_2 (a_1 Q_2 + a_0 Q_3 + \beta\Lambda) - P_2 Q_5$$

$$Q_3 Q_4 = Q_3 (a_1 Q_2 + a_0 Q_3 + \beta\Lambda) - P_3 Q_5$$

$$\Lambda Q_4 = \Lambda (a_1 Q_2 + a_0 Q_3 + \beta\Lambda) + (\delta X_2^2 + \gamma Y_2) Q_5$$

Furthermore the coefficient of Y_2^3 in $L(L+a_1 Q_3)P_1 + (\delta X_2^2 + \gamma Y_2)H_4$ is different from zero

Proof By (1.7) if A_1 and $F-A_1$ have common components, F contains an unique elliptic tail E, and $F = \sum_{i=1}^{n} A_i + D$ is a standard numerical decomposition, one of the following holds,in case (B):

(B_1) $n = 1$, $A_1 = E$ and E is not contained in $D-E$

(B_2) $n = 2$, $A_1 \supset A_2$, D is a 2-connected genus 1 divisor without common components with A_2, and $\mathcal{O}_E(A_1) \not\equiv \mathcal{O}_E$

(B_3) $n = 3$, $A_1 \supset A_2 \supset A_3$, $D = \mathbb{P}^1$ and $\mathrm{Im}\{S_1 \to H^0(D, K_F)\}$ has dim 1 $(\mathcal{O}_E(A_1) \not\cong \mathcal{O}_E$, $\mathcal{O}_E(A_1 + A_2) \cong \mathcal{O}_E)$

(B_4) $n = 3$, $A_1 \supset A_2 \supset A_3$, $D = \mathbb{P}^1$ and $S_1 \to H^0(D, K_F)$ is surjective $(\mathcal{O}_E(A_1) \not\cong \mathcal{O}_E$ and $\mathcal{O}_E(A_1 + A_2) \not\cong \mathcal{O}_E)$.

The restriction maps $\varphi_n : S_n \to H^0(E, nK_F)$ are surjective and $\dim \mathrm{Ker}\,\varphi_n = 2n-1$. The restriction maps $\psi_n : S_n \to H^0(F-A_1-E, nK_F)$ are such that $\mathrm{Ker}\,\psi_n = H^0(E, nK_F - (F-A_1-E)) = H^0(E, (n-1)K_F + A_1)$ and thus $\mathrm{Ker}\,\psi_1 = 0$. Furthermore if $F-A_1$ is 2-connected (case B_1) $\mathrm{Ker}\,\psi_2 = H^0(E, \mathcal{O}_E)$ and $\mathrm{Ker}\,\psi_n = H^0(E, (n-2)K_F)$ for $n \geq 3$. If $F-A_1$ is not 2-connected, then $\mathrm{Ker}\,\psi_n = H^0(E, (n-1)K_F + A_1)$ is $(n-1)$-dimensional for $n \geq 2$.

Let (x_1, x_2) be a basis of S_1 such that $x_{1|_E} = 0$, i.e. $x_1 \in \mathrm{Ker}\,\varphi_1$. Then $x_{2|_E}$ generates $H^0(E, K_F)$ and we choose x_2 such that x_2 is a generic element of S_1.

In degree 2 we have that $(x_1^2, x_1x_2) \in \mathrm{Ker}\,\varphi_2$. Let y_1, y_2 be the two new generators of S in degree 2. Since $2K_F$ is generated by its global sections we can choose y_2 such that x_2^2, y_2 form a free pencil in S_2.

Then $x_{2|_E}^2$, $y_{2|_E}$ generate $H^0(E, K_F)$. Since $\mathrm{Ker}\,\varphi_2$ is 3-dimensional we choose y_1 such that $y_1 \in \mathrm{Ker}\,\varphi_2$.

Now $\mathrm{Ker}\,\psi_2$ is 1-dimensional and thus there exists $t = \alpha y_2 + \beta x_2^2 + \gamma x_1^2 + \mu x_2x_1 + \delta y_1 \in \mathrm{Ker}\,\psi_2$. Since $x_{1|_E} = 0$ we have necessarily $x_1 t = 0$.

Applying the free pencil trick, we get that $\mathrm{Im}\{x_2^2 S_2 + y_2 S_2 \longrightarrow S_4\}$ has dimension 9 and so $(x_2^2x_1^2, x_2^3x_1, x_2^4, x_2^2y_1, x_2^2y_2, x_1^2y_2, x_1x_2y_2, y_2^2, y_1y_2)$ are independent elements of S_4. But then no relation of the type $\lambda_0 x_1y_2 + \lambda_1 x_1x_2^2 + \lambda_2 x_1^2x_2 = 0$ can hold in S and thus either y_1 or x_1^2 appear in t with nonzero coefficient. Changing y_1 by a multiple of x_1^2 if necessary we can then assume that x_1^2 appears in t with nonzero coefficient. Thus in degree 3 there is a relation

$$x_1^3 + \alpha x_1y_2 + \beta x_1x_2^2 + \mu x_1^2x_2 + \delta x_1y_1 = 0 .$$

Let us remark that if $E \subset F\text{-}A_1\text{-}E$ (cases B_3, B_4) then $\alpha = \beta = 0$ and if $E \not\subset F\text{-}A_1\text{-}E$, $\alpha \neq 0$ or $\beta \neq 0$. Thus we can write $t = x_1^2 + \delta x_2^2 + \gamma y_2 + {}$ $+ a_0 x_1x_2 + a_1 y_1$ where $\gamma \neq 0$ or $\delta \neq 0$ if E is not contained in $F\text{-}A_1\text{-}E$ (cases B_1, B_2) and $\delta = 0$, $\gamma = 0$ otherwise (cases B_3, B_4).

In case (B_4) since the restriction map $S_1 \to H^0(D, \mathcal{O}_D(1))$ is surjective, $a_1 \neq 0$. Changing y_1, if necessary, we can assume that $a_0 = 0$. In case (B_3), since $x_{2|C}$ depends on x_1, we have $a_0 \neq 0$ and $a_1 = 0$.

In case (B_2), because the restriction map $S_1 \to H^0(D, K_F)$ is surjective, the restrictions of x_1^2, x_1x_2, x_2^2 to D are independent (cf. I.7.3) and thus $\gamma \neq 0$.

By (1.17), $\mathrm{Im}\{S_1 \otimes S_2 \to S_3\}$ has codimension 1 and since we have $x_1 t = 0$, $\mathrm{Ker}\,\varphi_3$ has as a basis $x_1^2x_2, x_1x_2^2, x_1y_1, x_2y_1, x_1y_2$.

Let z be the new generator of S in degree 3 . Then the restrictions to E of x_2^3, x_2y_2, z form a basis of $H^0(E, 3K_F)$.

By the free pencil trick, $x_2^2 .S_3 + y_2.S_3 \to S_5$ is surjective and thus $(x_1^2x_2^3, x_1x_2^4, x_1x_2^2y_1, x_2^3y_1, x_1x_2^2y_2, x_1^2x_2y_2, x_1y_1y_2, x_2y_1y_2, x_1y_2^2)$ form a basis of $\operatorname{Ker}\varphi_5$.

But then $(x_1^2x_2^2, x_1x_2^3, x_1x_2y_1, x_2^2y_1, x_1x_2y_2, x_1^2y_2, y_1y_2) \in \operatorname{Ker}\varphi_4$ are independent and thus,since $\dim \operatorname{Ker}\varphi_4 = 7$, form a basis of $\operatorname{Ker}\varphi_4$.

Now $x_1z, x_1^2y_1, y_1^2$ belongs to $\operatorname{Ker}\varphi_4$ also. Since $y_1 \in \operatorname{Ker}\varphi_2$ we have $y_1t = 0$ and thus $(t = x_1^2 + \delta x_2^2 + \delta\gamma\, y_2 + a_0x_1x_2 + a_1y_1)$ a relation

$$U_1 : x_1^2y_1 + \delta\, x_2^2y_1 + \gamma\, y_1y_2 + a_0x_1x_2y_1 + a_1y_1^2 = 0 \text{ in } S .$$

. From $y_1^2 \in \operatorname{Ker}\varphi_4$, $x_1z \in \operatorname{Ker}\varphi_4$ we have then relations

$$U_0 : y_1^2 = b_0x_1^2x_2^2 + b_1x_1x_2^3 + b_2x_1^2y_2 + b_3x_1x_2y_2 + b_4x_1x_2y_1 + b_5\, x_2^2y_1 + y + b_6\, y_1y_2$$

$$U_2 : x_1z = c_0\, x_1^2x_2^2 + c_1x_1x_2^3 + c_2x_1^2y_2 + c_3x_1x_2y_2 + c_4x_1x_2y_1 + c_5x_2^2y_1 + c_6y_1y_2$$

Using the fact that $x_2^2\, S_2 + y_2\, S_n \to S_{n+2}$ is surjective for $n \geq 3$ and looking at the relations in S established so far, it is easy to see that in degree 5 there is only a new relation given by $y_1z \in \operatorname{Ker}\varphi_5$.

$$V : y_1 z = d_0 x_1^2 x_2^3 + d_1 x_1 x_2^4 + d_2 x_1 x_2^2 y_1 + d_3 x_2^3 y_1 +$$

$$+ d_4 x_1 x_2^2 y_2 + d_1 x_1^2 x_2 y_2 + d_6 x_1 y_1 y_2 + d_7 x_2 y_1 y_2 + d_8 x_1 y_2^2$$

Now in degree 6 we have (cf. description of $R(E, K_F)$ in (I.7.3))

$$z^2{}_{|E} = (\lambda\, y_2^3 + x_2^2\, P_4(x_2,y_2) + x_2 z_2 P_3(x_2,y_2))_{|E} \quad \text{with } \lambda \neq 0\,.$$

Since the map $x_2^2\, S_4 + y_2\, S_4 \to S_6$ is surjective, this relation lifts to a relation in S.

Looking at the relations established so far and using a dimension count we have, because $x_2^2\, S_n + y_2\, S_n \to S_{n+2}$ is surjective for $n \geq 4$, that there are no other independent relations in S.

Changing z to simplify U_2 we can then write the relations in S as

$$T : x_1^3 + x_1(\delta\, x_2^2 + \gamma\, y_2) + a_0 x_1^2 x_2 + a_1 x_1 y_1 = 0$$

$$U_0 : y_1^2 - x_1^2 p_1 - x_1 x_2 p_2 - y_1 p_3 - \beta\, x_1 x_2 y_1 = 0$$

$$U_1 : x_1^2 y_1 + y_1(\delta\, x_2^2 + \gamma\, y_2) + a_0\, x_1 x_2 y_1 + a_1 x_1^2 p_1 + a_1 x_1 x_2 p_2 + a_1 y_1 p_3 = 0$$

$$U_2 : x_1 z - y_1\, \ell = 0$$

$$V : y_1 z - x_1 f - x_2 y_1 q_1 - x_1^2 x_2 q_2 - x_1 y_1 q_3 = 0$$

$$W : z^2 - \lambda\, y_2^3 - x_2^2 h_1 - x_1 x_2 h_2 - y_1 h_3 - x_1^2 h_4 - x_2 z\, q_4 - x_1 x_2 y_1\, q_5 = 0$$

where $p_1, p_2, p_3, q_1, q_2, q_3, q_4, q_5, \ell$ are polynomials of degree 2 in $x_2, y_2,$

and f, h_1, h_2, h_3, h_4 are polynomials of degree 4 in x_2, y_2.

Now we are going to see how these polynomials are related.

Consider the sets A_n where $A_n = \left\{x_2^{\,n}, x_2^{\,n-2}y_2, \ldots, y_2^{\,n/2}\right\}$ if n is even or $A_n = \left\{x_2^{\,n}, x_2^{\,n-2}y_2, \ldots, x_2 y_2^{\,(n-1)/2}\right\}$ if n is odd.

For each n $(A_n, x_1A_{n-1}, x_1^2A_{n-2}, y_1A_{n-2}, zA_{n-3}, x_1y_1A_{n-3})$ forms a basis of S_n.

Let us remark that if γ in T is different from zero, we can, by changing y_2, assume that $\delta = a_0 = a_1 = 0$. The above set remains a basis.

Now, to deduce the relations between the coefficients above, we compare the various relations.

From $y_1 U_2$ we get

(1) $\quad x_1y_1z = y_1^2\,\ell$

and from $x_1 V$ we get

(2) $\quad x_1y_1z = x_1^2\, f + x_1x_2y_1q_1 + x_1^3x_2\, q_2 + x_1^2y_1\, q_3$.

Using the other relations in S, we can reduce the right-hand side of (1) and (2) to an expression on the elements of the basis described above and thus comparing (1) and (2) we get a linear relation between independent elements of S_n, giving identities between the coefficients of the polynomials. We omit the calculations since they are purely mechanical.

From comparing the expressions obtained from (1) and (2) we get (these are identities between the polynomials not relations in S)

$$\begin{cases} p_1(\ell + a_1 q_3) + a_0 x_2^2 q_2 = f \\ p_2(\ell + a_1 q_3) = -(\delta x_2^2 + \gamma y_2) q_2 \\ p_3(\ell + a_1 q_3) = -(\delta x_2^2 + \gamma y_2) q_3 \\ q_1 = a_1 q_2 + a_0 q_3 + \beta(\ell + a_1 q_3) \end{cases} .$$

Comparing the expressions obtained from $z U_2$ and $x_1 W$ we get

$$\begin{cases} \lambda y_2^3 + x_2^2 h_1 = f \ell + (\delta x_2^2 + \gamma y_2) h_4 + a_1 x_2^2 p_2 q_5 \\ (q_1 - q_4)\ell = -(\delta x_2^2 + \gamma y_2) q_5 - a_1 p_3 q_5 \\ h_2 = q_2 \ell + a_1 p_1 q_5 + a_0 h_4 \\ h_3 = q_3 \ell + (a_0 + a_1 \beta) x_2^2 q_5 + a_1 h_4 \end{cases}$$

and comparing the expressions obtained from $z U_0$ and $y_1 V$ we get

$$\begin{cases} p_2 \ell + p_3 q_1 = -(\delta x_2^2 + \gamma y_2)(q_2 + \beta q_3) + a_0 p_3 q_3 \\ p_3 f = -(\delta x_2^2 + \gamma y_2) p_1 q_3 + a_0 x_2^2 p_2 q_3 \\ p_3 q_2 = p_2 q_3 \end{cases} .$$

Comparing the expressions obtained from zV and $y_1 W$ we obtain

$$\begin{cases} (q_1 - q_4) f = a_0 x_2^2 p_2 q_5 - (\delta x_2^2 + \gamma y_2) p_1 q_5 \\ (q_1 - q_4) q_1 = a_1 p_2 q_5 + a_0 p_3 q_5 - \beta(\delta x_2^2 + \gamma y_2) q_5 \\ (q_1 - q_4) q_3 = p_3 q_5 \end{cases}$$

and thus we have

$$f = p_1(\ell + a_1 q_3) + a_0 x_2^2 q_2$$

$$q_1 = a_1 q_2 + a_0 q_3 + \beta(\ell + a_1 q_3)$$

$$h_2 = q_2 \ell + a_1 p_1 q_5 + a_0 h_4$$

$$h_3 = q_3\,\ell + (a_0 + a_1\beta)\,x_2^2\,q_5 + a_1\,h_4$$

$$\lambda\,x_2^3 + x_2^2\,h_1 = p_1\,\ell(\ell + a_1\,q_3) + a_0\,x_2^2\,q_2\,\ell + (\delta\,x_2^2 + \gamma\,y_2)h_4 + a_1\,x_2^2\,p_2 q_5$$

and identities between polynomials

$$p_2\,(\ell + a_1\,q_3) = -\,(\delta\,x_2^2 + \gamma\,y_2)\,q_2$$

$$p_3\,(\ell + a_1\,q_3) = -\,(\delta\,x_2^2 + \gamma\,y_2)\,q_3$$

$$p_2\,q_3 = p_3\,q_2$$

$$(q_1 - q_4)\,(\ell + a_1\,q_3) = -\,(\delta\,x_2^2 + \gamma\,y_2)\,q_5$$

$$(q_1 - q_4)\,q_2 = p_2\,q_5$$

$$(q_1 - q_4)\,q_3 = p_3\,q_5$$

Since $\lambda \neq 0$, if $\gamma = 0$ necessarily the coefficients of y_2 in p_1, ℓ, $(\ell + a_1\,q_3)$ are different from zero.If $\gamma \neq 0$, then U_1 is $x_1^2 y_1 + \gamma\,y_1 y_2 = 0$. Then if $\gamma \neq 0$ we have $x_1 z = \alpha_0\,x_2^2\,y_1 + \alpha_1 y_1 y_2$ and $x_1^2\,y_1 + \gamma\,y_1 y_2 = 0$. If $\alpha_0 = 0$, we have $x_1\,(z + \frac{1}{\gamma} x_1 y_1) = 0$ and thus $\operatorname{Ker}\psi_3$ is 2-dimensional and thus we are in case (B_2). If $\gamma \neq 0$ and $\alpha_0 \neq 0$ we are in case (B_1).

Rewriting the relations using the identities above we can present S as stated.

End of proof of (4.5)

(4.6) Proposition If F is analytically of type I and F contains an elliptic tail appearing with multiplicity larger than 1 in F, $R(F, K_F)$ can be presented as $k[X_0, X_1, X_2, Y_1, Y_2, Z] / I$ where I is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} 0 & X_0 & X_1 & Y_1 \\ X_0 & A & Y_2 & Z \end{pmatrix}$$

and the polynomials

$$U' : Y_1(Y_1-\alpha' X_1X_2-\varepsilon X_1^2-\mu X_0X_2) - \lambda X_1^2P_1 - \lambda X_1X_2P_2 - X_0X_2Q_2$$

$$V' : Z(Y_1-\alpha' X_1X_2-\varepsilon X_1^2-\mu X_0X_2) - \lambda X_1Y_2P_1 - \lambda X_2Y_2P_2 - X_2Q_2A -$$

$$- X_0(H_4 - a_1\mu X_2^2 Q_2)$$

$$W' : Z(Z-\alpha' X_2Y_2-\varepsilon X_1Y_2-\mu X_2 A) - \lambda Y_2^2P_1 - A(H_4-a_1 \mu X_2^2Q_2)$$

$$- X_2Q_2 (X_1Y_2 + a_0X_2Y_2 + a_1Z) - X_0X_2S$$

where

$$A = X_1^2 + \delta X_2^2 + \gamma Y_2 + a_0 X_1X_2 + a_1Y_1,$$

$$P_1, P_2, Q_2 \in k[X_2, Y_2]_2 \ , \ H_4, S \in k[X_2, Y_2]_4,$$

$$\lambda = 1 + a_1 \varepsilon \neq 0, \quad \lambda Y_2P_2 = -(\delta X_2^2 + \gamma Y_2)Q_2 .$$

The coefficients δ, γ, a_0, a_1 are such that one of them is nonzero and $\gamma a_0 = \gamma a_1 = a_0 a_1 = 0.$

Proof By (1.2) we can use the techniques of (II.1) to recover R from S (for details see the construction of R for type III fibres in section 2).

F contains a unique elliptic tail E, with $E \subset F-A_1$. The ring S is described in (4.5) and $X_1, X_2, Y_1, Y_2, Z_2 \in S$ are such that $x_{2|_E}, y_{2|_E}, z_{2|_E}$ generate $R(E, K_F)$ and $x_{1|_E} = y_{1|_E} = 0$.

Then we have $R = k[X_0, X_1, X_2, Y_1, Y_2, Z] / \tilde{I}$ where $x_0 \in R_1$ is such that $x_{0|_{F-E}} = 0$, and by the construction of S, $J_R = \{ z \in R : x_0 s = 0\} = $ $=(x_0, x_1, y_1)$.

Thus, denoting $T_0 = X_1 A$, $T_1 = Y_1 A$, $T_2 = X_1 Z - Y_1 L$, we have $\tilde{I}$ generated by

$$X_0^2$$
$$X_0 X_1$$
$$X_0 Y_1$$
$$\tilde{T}_0 = T_0 + X_0 N_0$$
$$\tilde{T}_1 = T_1 + X_0 N_1$$
$$\tilde{T}_2 = T_2 + X_0 N_2$$
$$\tilde{U} = U + X_0 U_0$$
$$\tilde{V} = V + X_0 V_0$$
$$\tilde{W} = W + X_0 W_0$$

where $N_0, N_1, N_2, U_0, V_0, W_0 \in k[X_2, Y_2] + Z\, k[X_2, Y_2]$.

To find how these polynomials look like, because of the nilpotent structure of x_0, we will have to use syzygies (for details see similar constructions in (2.4) (a) or (2.1) (a) or (3.3) Step 3).

The generators of I satisfy the following syzygies

$$M_1 : Y_1 T_0 - X_1 T_1$$
$$M_2 : A T_2 + L T_1 - Z T_0$$

$M_3 : X_1V - (L + a_1q_3)U - Y_1T_2 + X_2Q_2T_0 + Q_3T_1$

$M_4 : (Y_1-P_3)V + X_2Q_2T_1 - (Z - (X_1+a_0X_2)Q_3)U -$

$- (X_1P_1 + X_2P_2 + \beta X_2Y_1)T_2 + P_1Q_3T_0 + \beta X_2Q_3T_1$

$M_5 : X_1W + H_4T_0 - (Z-X_1Q_3-X_2Q_4)T_2 - LV + X_2Q_5T_1 - a_1X_2Q_5U$

$M_6 : AU-(Y_1-P_3)T_1+(X_1P_1+X_2 P_2 + \beta X_2Y_1)T_0$

$M_7 : AV - (Z - (X_1+a_0X_2)Q_3 - \beta X_2(L+a_1Q_3) - a_1X_2Q_2)T_1+$

$+ (P_1(L + a_1 Q_3) + (X_1 + a_0 X_2)X_2 Q_2)T_0$

$M_8 : (Z + a_0X_2Q_3 + a_1X_2Q_2 - X_2Q_4 + \beta X_2(L+a_1Q_3))V - Y_1W-$

$- (H_4+\beta X_2^2 Q_5)T_1 + ((L+a_1Q_3)P_1 + (X_1 + a_0X_2)X_2Q_2)T_2 -$

$- ((X_1 + a_0X_2)X_2 Q_5)U - X_2P_1Q_5T_0 .$

We will first make a coordinate change so that $L = Y_2$. If the coefficient of Y_2 in L is different from zero we can assume by changing Y_2 that $L = Y_2$. We have seen that this coefficient can only be zero if $\gamma \neq 0$. If it is zero we then have

$$T_1 = X_1^2 Y_1 + \gamma Y_1Y_2$$

$$T_2 = X_1Z - \alpha_0 X_2^2Y_1 , \text{ and so}$$

$$\frac{1}{\gamma}T_1 + T_2 = X_1Z + \frac{1}{\gamma}X_1^2Y_1 - \alpha_0 X_2^2Y_1 - Y_1Y_2 .$$

Thus modifying Y_2 and Z we can also assume that $L = Y_2$. Remark that S keeps the same presentation and syzygies. The only thing that changes is the condition $\delta\gamma = 0$, that does not longer hold. We will then have

$A = X_1^2 + \delta X_2^2 + \gamma Y_2$ with $\delta \neq 0$ in case (B_1) and $\delta = 0$ in case (B_2).

If $\gamma \neq 0$ we can also change Y_1 (and Z accordingly) by X_1X_2 so that P_1

in U is nonzero.

Now $\tilde{T}_2$ is $X_1Z - Y_1Y_2 + \gamma_0 X_0X_2^3 + \gamma_1 X_0X_2Y_2 + \gamma_2 X_0Z$ and changing X_1, Y_1 by multiples of X_0 we can assume that $\gamma_1 = \gamma_2 = 0$.

Then from M_2 we obtain

$$X_0\big((\delta X_2^2 + \gamma Y_2)\gamma_0 X_2^3 + Y_2 N_1 - Z N_0\big) \in \tilde{I}$$

and so (because $\tilde{I} \cap X_0(k[X_2, Y_2] + Z\,k[X_2, Y_2]) = 0\big)$

$$\begin{cases} N_1 &= \alpha Z - \gamma\gamma_0 X_2^3 \\ N_0 &= \alpha Y_2 \\ \delta\gamma_0 &= 0 \end{cases}$$

If $\alpha = 0$ we would have $x_1 a = 0$ in R and thus $x_1 \in \operatorname{Ker}(H^0(F, K_F) \to H^0(2E, K_F))$. In case (B_1) this kernel is zero and in cases (B_3), (B_4) this kernel is 1-dimensional and contains x_0. Thus in cases (B_1), (B_3), (B_4), $\alpha \neq 0$. In case (B_2) this kernel is 1-dimensional generated by $t' = x_1 + \varepsilon x_0$ and thus $\operatorname{Ker}(H^0(F, 2K_F) \to H^0(2E, 2K_F))$ which is 2-dimensional is generated by $x_1 t'$, $x_2 t'$. So Y_1 does not belong to this kernel and $N_1 \neq 0$.

Then from M_3 we obtain

$$Q - Y_2U_0 - a_1Q_3U_0 + \alpha Y_2X_2Q_2 + Q_3(\alpha Z - \gamma\gamma_0 X_2^3) \equiv 0$$

giving

$$\begin{cases} Q_3 = \varepsilon Y_2 \\ (1+a_1\varepsilon)U_0 = (+\alpha X_2Q_2 + \varepsilon(\alpha Z - \gamma\gamma_0 X_2^3)) \end{cases}.$$

Now we have seen that if $\gamma = 0$, $1 + a_1\varepsilon \neq 0$ and if $\gamma \neq 0$ by the choice of Y_2, $a_1 = 0$. So $1 + a_1\varepsilon \neq 0$ and thus

$$U_0 = \frac{1}{1+a_1\varepsilon}\left(+\alpha X_2 Q_2 + \varepsilon(\alpha Z + \gamma\gamma_0 X_2^3)\right) .$$

Then (see how S is presented) we have

$$P_3 = -\frac{\varepsilon}{1+a_1\varepsilon}(\delta X_2^2 + \gamma Y_2)$$

and

$$\text{for } \delta \neq 0 \quad \begin{cases} Q_5 = \mu (1+a_1\varepsilon)Y_2 \\ Q_4 = a_1 Q_2 + (a_0\varepsilon + \beta(1+a_1\varepsilon))Y_2 + \mu(\delta X_2^2 + \gamma Y_2) \\ Q_2 = \nu(1+a_1\varepsilon)Y_2 \\ P_2 = -\gamma(\delta X_2^2 + \gamma Y_2) \end{cases}$$

$$\text{for } \delta = 0 \quad \begin{cases} Q_5 = \mu_0(1+a_1\varepsilon)X_2^2 + \mu(1+a_1\varepsilon)Y_2 \\ Q_4 = a_1 Q_2 + (a_0\varepsilon + \beta(1+a_1\varepsilon))Y_2 + \frac{\gamma}{1+a_1\varepsilon} Q_5 \\ P_2 = -\frac{\gamma}{1+a_1\varepsilon} Q_2 \end{cases}$$

Now from the syzygy M_5 we then get

$$\begin{cases} \alpha\mu_0 = \gamma_0 \\ V_0 = \alpha H_4 + \mu\alpha X_2 Z - a_1 \mu X_2^2 Q_2 + \gamma_0(a_0\varepsilon+\beta(1+a_1\varepsilon))X_2^4 \end{cases}$$

Thus $\alpha \neq 0$ in case (B_2) too (otherwise $N_1 = 0$) and we can assume that $\alpha = -1$.

From M_7 we get for $\delta = 0$

$$\begin{cases} \beta\mu\gamma\gamma_0 = 0 \\ \gamma_0 X_2 Y_2^2 P_1 - \gamma_0 X_2^3 Y_2 P_1 = 0 \end{cases}$$

Since $P_1 \neq 0$ we then have $\gamma_0 = 0$ (and $\mu_0 = 0$). The other syzigies give no further information.

The presentation of $R(F, K_F)$ as in the statement is obtained by letting :

$\alpha' = (a_0\varepsilon + \beta(1+a_1\varepsilon))$,

$U' = (1+a_1\varepsilon)\left(U - \frac{\varepsilon}{1+a_1\varepsilon}T_1\right) - \mu\, X_2(X_0Y_1)$,

$V' = V + (\varepsilon X_1 + \alpha X_2)T_2$

$W' = W - a_1\, \mu\, X_2\, V'$.

Using the fact that changing Z by multiples of X_0 does not alter the other polynomials, we can assume that W'_0 has no terms in Z.

Section 5. The canonical ring of a double fibre.

(5.1) Theorem If $F = 2C$ is a double fibre of genus 3, then $R(F, K_F)$ can be presented as

$$k[X_0, X_1, X_2, Y_0, Y_1, Y_2, Z_1, Z_2] / I$$

where I is the ideal generated by the 20 polynomials in Table V-1 with

(i) $\delta \neq 0$, $\lambda \neq 0$ if C is 2-connected.

(ii) $\delta = 0$ if C is not 2-connected and $\lambda = 0$ if C contains a unique elliptic tail appearing with multiplicity 1 , $\lambda \neq 0$ if C contains two distinct elliptic tails.

Table V-1

X_0^2

X_0X_1

X_1^2

X_0Y_0

X_1Y_0

$X_0(Y_1 - \lambda Y_2) - \delta X_1X_2^2$

$X_1Y_1 - X_0Y_2$

Y_0^2

$X_0(Z_1 - \lambda Z_2) - \delta X_2^2Y_0$

$X_1Z_1 - X_0Z_2$

$Y_0Y_1 - X_0Z_2$

$Y_0Y_2 - X_1Z_2$

$Y_1(Y_1 - \lambda Y_2) - \delta X_2^2Y_2 - P_0$

$Y_2Z_1 - Y_1Z_2 - P_1$

$Y_1(Z_1 - \lambda Z_2) - \delta X_2^2Z_2 - P_2$

$Y_0Z_1 - X_0 Q$

$Y_0Z_2 - X_1 Q$

$Z_1(Z_1 - \lambda Z_2) - \delta X_2^2 Q - P_3$

$Z_1Z_2 - Y_1Q - P_4$

$Z_2^2 - Y_2 Q - P_5$

where $$Q = \delta_0 Y_2^2 + \delta_1 Y_1 Y_2 + \delta_2 X_2^2 Y_2 + \delta_3 X_2^2 Y_1 + \delta_4 X_2^4$$

with $\delta_0 \neq 0$

and $P_0, P_1, P_2, P_3, P_4, P_5 \in \left(X_2^{n-2} Y_0\right) + X_1 k[X_2, Y_2, Z_2] + X_0 k[X_2, Y_2, Z_2]$.

Proof This comes from combining (5.2) and (5.3) below

Remark There are various relations between the coefficients of P_1, P_2, P_3, P_4, P_5, but I have not completed the calculations.

(5.2) Proposition If $F = 2C$ is a double fibre of genus 3 and C is not 2-connected $R(F, K_F)$ can be presented as $k[X_0, X_1, X_2, Y_0, Y_1, Y_2] / I$ where I is the ideal generated by the 20 polynomials in Table V-1, with $\delta = 0$. If C contains an elliptic tail appearing with multiplicity 2 in C, $\lambda = 0$, and if C contains two distinct elliptic tails $\lambda \neq 0$.

Proof Let φ_n be as in (II.6.3) the restriction maps $\varphi_n : R_n \to H^0(C, nK_F)$ and let E, E' be the elliptic tails contained in C (see II.6.5). By (II.6.3) $\operatorname{Ker} \varphi_n = H^0(C, K_C + (n-1) K_F)$ and by (II.6.6), (II.6.7) we can decompose $\operatorname{Ker} \varphi_n$ and $\operatorname{Im} \varphi_n$ as (same notation as in II.6.6)

$$0 \to \operatorname{Ker} g_n = H^0(E, (n-1)K_F) \to \operatorname{Ker} \varphi_n \to \operatorname{Im} g'_n = H^0(E', (n-1)K_F + K_C) \to 0$$

and

$$0 \to \operatorname{Ker} f_n = H^0(E, (n-1)K_F + C) \to \operatorname{Im} \varphi_n \to \operatorname{Im} f'_n = H^0(E', nK_F) \to 0 .$$

Then we can consider the following filtration of R_n

$$0 \subset H^0(E, (n-1)K_F) = T_n \subset \operatorname{Ker} \varphi_n \subset \operatorname{Ker} \varphi_n + \operatorname{Ker} f_n = V_n \subset R_n .$$

We have

$$\dim T_n = n-1$$

$$\dim \operatorname{Ker} \varphi_n = \begin{cases} 2 & \text{for } n = 1 \\ 2n-1 & \text{for } n \geq 2 \end{cases}$$

$$\dim V_n = \begin{cases} 2 & \text{for } n = 1 \\ 3n-2 & \text{for } n \geq 2 \end{cases}$$

Remark that, since $F = 2C$ if $u \in \operatorname{Ker} \varphi_n$, $v \in \operatorname{Ker} \varphi_m$, we have $uv = 0$ in R.

Let us recall that E may be equal to E' and that by (II.6.4) the restriction maps

$$\rho_n : R_n \longrightarrow H^0(E, nK_F)$$
$$\rho'_n : R_n \longrightarrow H^0(E', nK_F)$$

are surjective for every $n \geq 1$. We have $\rho'_n = f'_n \circ \varphi_n$.

We will use the filtration above and the decompositions of $\operatorname{Ker} \varphi_n$, $\operatorname{Im} \varphi_n$ to find a presentation for $R(F, K_F)$.

By (II.6.6) $\operatorname{Ker} g_1$ is generated by a single element x_0 and the maps $\operatorname{Ker} g_1 \otimes H^0(E, nK_F) \longrightarrow \operatorname{Ker} g_{n+1}$ are surjective. Also, by (II.6.6) and (II.6.8) $\operatorname{Im} \{\operatorname{Ker} f_2 \otimes H^0(E, nK_F) \longrightarrow \operatorname{Ker} f_{n+2}\}$ has codimension 1 and for $n \geq 2$ a complementary element for this image will be equal to $\mu\varepsilon^{n-1}$, where $\varepsilon \in H^0(E, K_F)$, $\mu \in \operatorname{Ker} f_3 \setminus \operatorname{Im}\{\operatorname{Ker} f_2 \otimes H^0(E, K_F) \longrightarrow \operatorname{Ker} f_3\}$. Again by (II.6.8) $\operatorname{Im}\{\operatorname{Im} g_1 \otimes H^0(E', nK_F) \longrightarrow \operatorname{Im} g_{n+1}\}$ has codimension 1 and for $n \geq 2$ a complementary element for this image will be $\tau\varepsilon'^{n-1}$ where $\varepsilon' \in H^0(E, K_F)$ and $\tau \in \operatorname{Ker} g_2 \setminus \operatorname{Im}\{\operatorname{Ker} g_1 \otimes H^0(E', K_F) \longrightarrow \operatorname{Ker} g_2\}$. For the generation of $\operatorname{Im} f_n$ recall (I.7.2).

Let (x_0, x_1, x_2) be a basis of R_1 chosen such that $x_0 \in T_1$,

$x_1 \in \operatorname{Ker} \varphi_1$. Then, because ρ_1, ρ'_1 are surjective, $x_{2|_E}$ generates $H^0(E, K_F)$ and if $E \neq E'$, $x_{2|_{E'}}$ also generates $H^0(E', K_F)$.

Then in degree 2 we have $x_0^2 = x_0 x_1 = x_1^2 = 0$, and because x_2 generates $\operatorname{Im} \rho_1$, $\operatorname{Im} \rho'_1$ we have

$$(x_0 x_2) = T_2 \text{ and } x_1 x_2 \in \operatorname{Ker} \varphi_2 \setminus T_2 .$$

Let y_0 be a complementary element of (x_0x_2, x_1x_2) in $\operatorname{Ker} \varphi_2$. Then in the decomposition of $\operatorname{Ker} \varphi_2$ considered above (x_1x_2, y_0) generate $\operatorname{Im} g'_2$.

We will have two other generators $y_1, y_2 \in R_n \setminus \operatorname{Ker} \varphi_n$ and we can assume that $y_1 \in V_n$. Then $\operatorname{Im} \rho'_2$ is generated by x_2^2, y_2. If $E \neq E'$, $y_{1|_E} \neq 0$, and if $E = E'$, $y_{1|_{E'}} = 0$. By a convenient choice of y_2 we can assume that x_2^2, y_2 also generate $\operatorname{Im} \rho_2$. Then (x_2^2, y_2) have no common zeros, and $y_{1|_E} = \lambda\, y_{2|_E}$, with $\lambda = 0$ if $E = E'$, $\lambda \neq 0$ if $E \neq E'$.

In degree 3 we have $x_0 y_0 = x_1 y_0 = 0$ and because $(y_1 - \lambda\, y_2)_{|E} = 0$, $x_0(y_1 - \lambda\, y_2) = 0$.

Now x_2^2, y_2 generate $\operatorname{Im} \rho_n, \operatorname{Im} \rho'_n$ and so $(x_0x_2^2, x_0y_2, x_1x_2^2, x_1y_2) \in \operatorname{Ker} \varphi_3$ and $(x_2y_1, x_2^3, x_2y_2) \in R_3 \setminus \operatorname{Ker} \varphi_3$ are independent.

As $y_{1|_{E'}} = 0$ we will have $x_1 y_1 \in T_3$ and thus since $\operatorname{Ker} \varphi_3$ is generated by $x_0x_2^2, x_0y_2$ we will have $x_1y_1 = \varepsilon_0\, x_0x_2^2 + \varepsilon_1\, x_0y_2$.

Since $y_1 \in R_n \setminus \operatorname{Ker} \varphi_n$ necessarily ε_0 or ε_1 are different from zero. Now clearly (x_1x_2, y_0) generate $\operatorname{Im} g'_2$.

Let P be the point such that $\mathcal{O}_{E'}(K_F)=\mathcal{O}_E(P)$. Since $\mathcal{O}_{E'}(K_C)\neq\mathcal{O}_{E'}(K_F)$, the unique zero of $x_1 \in \mathrm{Im}\, g'_2$ will be a point $Q \neq P$. As $\mathrm{Im}\, g'_2$ is generated by its global sections $x_1x_2^2, x_2y_0$ are independent in $\mathrm{Im}\, g'_3$. But $x_1y_2 \in \mathrm{Im}\, g'_3$ is such that $x_1y_2(P) \neq 0$ and thus $x_1x_2^2, x_2y_0, x_1y_2$ are independent in $\mathrm{Im}\, g'_3$.

Thus $(x_0x_2^2, x_0y_2, x_1x_2^2, x_2y_0, x_1y_2)$ form a basis of $\mathrm{Ker}\, \varphi_3$.

There are two new generators $z_1, z_2 \in R_3 \setminus \mathrm{Ker}\, \varphi_3$ and as in degree 2 we can choose $z_1 \in V_n$, and z_2 such that (x_2^3, x_2y_2, z_2) generate $\mathrm{Im}\, \rho_n, \mathrm{Im}\, \rho'_n$. Then again, choosing z_1, we can assume that $z_{1|E'} = \lambda z_2$ with $\lambda = 0$ if $E = E'$, $\lambda \neq 0$ otherwise.

In degree 4, looking again at the identifications above we will have a basis for T_4 given by $x_0x_2^3, x_0x_2y_2, x_0z_2$ and relations

$$x_0(z_1-\lambda z_2) = 0\,,\quad x_1z_1 = a_0x_0x_2^3 + a_1x_0x_2y_2 + a_2x_0z_2$$

with $a_0 \neq 0$ or $a_1 \neq 0$ or $a_2 \neq 0$. Since $y_0 \in \mathrm{Ker}\, \varphi_2$, we have $y_0^2 = 0$.

By the free pencil trick the map $H^0(E', 2K_F) \otimes H^0(E', K_F + K_C) \to H^0(E', 3K_F + K_C)$ is surjective and $(x_1x_2^3, x_2^2y_0, y_0y_2, x_1x_2y_2)$ form a basis for $\mathrm{Im}\, g'_4$. Since x_1z_2 also belongs to $\mathrm{Im}\, g'_4$ and $x_1(P) \neq 0$, $z_2(P) \neq 0$ we have in $\mathrm{Im}\, g'_4$

$$x_1z_2 = \lambda\, y_0y_2 + c_0\, x_1x_2^3 + c_1\, x_2^2y_0 + c_2x_1x_2y_2\,,\ \text{with } \lambda \neq 0.$$

Then in R_4 we have

$$x_1z_2-\lambda\, y_0y_2 - c_0x_1x_2^3-c_1x_2^2y_0-c_2x_1x_2y_2 \in T_4$$

and thus a relation given by the fact that $T_4 = \langle x_0x_2^3, x_0x_2y_2, x_0z_2\rangle$.

Thus $(x_0x_2^3, x_0x_2y_2, x_0z_2, x_1x_2^3, x_1x_2y_2, x_2^2y_0, y_0y_2)$ is a basis of $\operatorname{Ker}\varphi_4$. Now y_0y_1 also belongs to T_4 and thus we get another relation.

It is easy to see now (again using the decomposition of $\operatorname{Im}\varphi_4$ and the fact that $\mathcal{O}_E\big((n-1)K_F + C\big) \not\cong \mathcal{O}_E$ that $(x_2^2y_1, x_2z_1, y_1y_2, x_2^4, x_2^2y_2, x_2z_2, y_2^2)$ is a basis of $\operatorname{Im}\varphi_4$ and we will have only another relation given by

$y_1(y_1-\lambda\, y_2) \in \operatorname{Ker}\varphi_4$.

Now in degree 5 we have

$$T_5 = \langle x_0x_2^4, x_0x_2^2y_2, x_0y_2^2, x_0z_2\rangle$$

$$\operatorname{Ker}\varphi_n = T_5 + \langle x_1x_2^4, x_1x_2^2y_2, x_1x_2z_2, x_1y_2^2, y_0x_2^3\rangle$$

and thus, since $y_0z_1 \in T_5$, $y_0z_2 \in \operatorname{Ker}\varphi_5$ we get two new relations. Because the map $\operatorname{Im} g'_2 \otimes H^0(E', 3K_F) \longrightarrow \operatorname{Im} g''_5$ is surjective and x_1z_2 depends on y_0y_2 necessarily the coefficient of $x_1y_2^2$ in the relation involving y_0z_2 must be different from zero.

Now, by (II.6.8), $(x_2^3y_1, x_2y_1y_2, y_1z_2, x_2^2z_1)$ form a complementary basis of $\operatorname{Ker}\varphi_n$ in V_n, and thus because $y_2z_1 \in V_n$ we will have a new relation.

Since by the free pencil trick the map $\operatorname{Ker} f_3 \otimes H^0(E, 2K_F) \longrightarrow \operatorname{Ker} f_5$ is surjective necessarily

$$y_2 z_1 = \gamma\, y_1 z_2 + \ldots \qquad , \text{with } \gamma \neq 0 .$$

There is only another relation in degree 5 given by

$$y_1(z_1 - \lambda\, z_2) \in \operatorname{Ker} \varphi_5 .$$

In the same way, in degree 6 we have

$$T_6 = \langle x_0 x_2^5, x_0 x_2^3 y_2, x_0 x_2 y_2^2, x_0 x_2^2 z_2, x_0 y_2 z_2 \rangle$$

$$\operatorname{Ker} \varphi_6 = T_6 + \langle x_1 x_2^5, x_1 x_2^3 y_2, x_1 x_2 y_2^2, x_1 x_2^2 z_2, x_1 y_2 z_2, x_2^4 y_0 \rangle$$

and
$$V_6 = \operatorname{Ker} \varphi_6 + \langle x_2^4 y_1, x_2^2 y_1 y_2, x_2 y_1 z_2, y_1 y_2^2, x_2^3 z_1 \rangle$$

where the elements appearing above are linearly independent.

Then we have a relation given by

$$z_1(z_1 - \lambda\, z_2) \in \operatorname{Ker} \varphi_6$$

and another given by

$$z_1 z_2 \in V_6 .$$

Using the fact that $\operatorname{Ker} g_3 \otimes H^0(E', 3K_F) \longrightarrow \operatorname{Ker} g_6$ is surjective it is easy to see that $y_1 y_2^2$ appears with nonzero coefficient in this last relation.

Also (c.f. I.7.2) $z^2_{2|E'} = \delta\, y_2^3 + P(x_2, y_2, z_2)$ with $\delta \neq 0$ and thus we have another relation given by $z^2 - \delta\, y_2^3 - P(x_2, y_2, z) \in V_6$.

Now counting monomials and dimensions and using the fact that $H^0(F, 2K_F) \otimes H^0(F, nK_F) \to H^0(F, (n+2)K_F)$ is surjective it is easy to see that there are no more independent relations.

So we have relations in R

$R_1 : x_0^2 = 0$

$R_2 : x_0 x_1 = 0$

$R_3 : x_1^2 = 0$

$R_4 : x_1 y_0 = 0$

$R_5 : x_0 y_0 = 0$

$R_6 : x_0(y_1 - \lambda\, y_2) = 0$

$R_7 : x_1 y_1 = x_0\, p_7$ (with $p_7 \neq 0$)

$R_8 : x_0(z_1 - \lambda\, z_2) = 0$

$R_9 : x_1 z_1 = x_0\, p_9$

$R_{10} : y_0^2 = 0$

$R_{11} : y_0 y_1 = x_0\, p_{11}$ (with $p_{11} \neq 0$)

$R_{12} : y_0 y_2 = \delta_0 x_1 z_2 + c_0 x_2^2 y_0 + x_1 x_2\, q_{12} + x_0\, p_{12}$ (with $\delta_0 \neq 0$)

$R_{13} : y_1(y_1 - \lambda\, y_2) = b_0 x_1 z_2 + d_0\, x_2^2 y_0 + x_1 x_2\, q_{13} + x_0\, p_{13}$

$R_{14} : y_1 z_2 = \delta_1\, y_2 z_1 + x_2 y_1\, f_{14} + d_1\, x_2^2 z_1 + d_2\, x_2^3 y_0 + 0\, x_1\, q_{14} + x_0\, p_{14}$ (with $\delta_1 \neq 0$)

$R_{15} : y_0 z_1 = x_0\, p_{15}$

$R_{16} : y_0 z_2 = \delta_2\, x_1 y_2^2 + d_3\, x_2^3 y_0 + x_1 x_2\, q_{16} + x_0\, p_{16}$ (with $\delta_2 \neq 0$)

$R_{17} : y_1\,(z_1 - \lambda\, z_2) = \varepsilon_0\, x_2^3 y_0 + x_1\, q_{17} + x_0\, p_{17}$

$R_{18} : z_1\,(z_1 - \lambda\, z_2) = \varepsilon_1\, x_2^3 y_0 + x_1\, q_{18} + x_0\, p_{18}$

$R_{19} : z_1z_2 = \gamma\, y_1y_2^2 + \varepsilon_2\, x_2^3z_1 + x_2y_1\, f_{19} + \varepsilon_3x_2^4y_0+x_1\, q_{19} + x_0\, p_{19}$ (with $\gamma \neq 0$)

$R_{20} : z_2^2 = \delta\, y_2^3 + x_2\, g + y_1\, f_{20} + \varepsilon\, x_2^3z_1 + \alpha x_2^4y_0 + x_1\, q_{20} + x_0\, p_{20}$ (with $\delta \neq 0$)

where $p_i, q_j, f_k, g \in k[x_2, y_2, z_2] \subset R$.

For each n , let A_n be the set of all monomials of degree n in x_2, y_2, z_2 . Then the set

$$\{x_0\, A_{n-1}, x_1\, A_{n-1}, y_1\, A_{n-2}, x_2^{n-2}\, y_0, x_2^{n-3}\, z_1, A_n\}$$

is a basis of R_n , for each $n \geq 3$.

Now we are going to find some of the relations between the coefficients of the polynomials above.

To simplify the calculations we make some coordinate changes. We can change z_2 so that the only monomial in R_{20} involving z_2 is z_2^2 and then if $\lambda \neq 0$, change z_1 accordingly so that $R_8, R_{13}, R_{17}, R_{18}$ still look the same.

Then we change y_0 so that $\delta_0 = 1$ and q_{12} , p_{12} have no monomials involving y_2 , and we change x_1 so that p_{12} has no term in z_2 .

If $\lambda = 0$, we change also y_2 so that c_0 in R_{12} is zero, and z_2 so that $\delta_1 = 1$.

Now from R_7 we have in R

(1) $x_1\, y_1\, z_2 = x_0\, z_2\, p_7$

and from R_{14} , R_9 , R_5 , R_1 , R_2 we have

(2) $x_1\, y_1\, z_2 = x_0\, (y_2 + d_1\, x_2^2)p_9 + x_0\, x_2\, f_{14}\, p_7$

giving

$$x_0(y_2 + d_1 x_2^2)p_9 - x_0(z_2 - x_2 f_{14})p_7 = 0 .$$

This is a linear relation between independent elements of R and thus

$$(y_2 + d_1 x_2^2)p_9 = (z_2 - x_2 f_{14})p_7 .$$

In a similar way we obtain

$$\lambda z_2 p_7 = \lambda y_2 p_9$$

$$(y_2 - c_0 x_2^2)p_{11} = (z_2 + x_2 q_{12})p_7 + \lambda y_2 p_{12} .$$

Analyzing these relations we obtain

$$\text{for } \lambda \neq 0 \begin{cases} p_7 = \gamma_1 y_2 \\ p_9 = \gamma_1 z_2 \\ c_0 = 0 \\ f_{14} = 0 \\ p_{11} = \gamma_1(z_2 + x_2 q_{12}) + \lambda p_{12} \\ \delta_1 = 1 \\ d_1 \gamma_1 = 0 \end{cases}$$

and

$$\text{for } \lambda = 0 \begin{cases} d_1 \gamma_1 = 0 \\ p_7 = \gamma_1 y_2 \\ p_9 = \gamma_1 z_2 - x_2 f_{14} \\ p_{11} = \gamma_1(z_2 + x_2 q_{12}) \end{cases} .$$

Since $p_7 \neq 0$ we must have $\gamma_1 \neq 0$ and so $d_1 = 0$. We can assume that $\gamma_1 = 1$.

Letting

$$q_{12} = a_0 x_2^2$$

$$p_{12} = a_1 x_2^3$$

we then have

$$p_{11} = z_2 + \mu\, x_2^3 \ \text{ with } \ \mu = a_0 + \lambda\, a_1 .$$

Let g in R_{20} be $g = x_2^2\, g_1(x_2, y_2)$. Comparing R_{11} and R_{14} we obtain

$$x_0\, z_2\, p_{11} - x_0\, y_2\, p_{15} = 0 \ \text{ i.e.}$$

$$x_0(z_2^2 + \mu\, x_2^3\, z_2) - x_0\, y_2\, p_{15} = 0 .$$

Subtracting $x_0 R_{20}$ from this last expression we obtain a linear relation between independent elements of R_7 that gives

$$\begin{cases} g_1 = c_0\, x_2^2\, y_2 + c_1\, y_2^2 = y_2\, g'_1 \\ \mu + \lambda\, \varepsilon = 0 \\ p_{15} = \delta\, y_2^2 + x_2^2\, g'_1 + \lambda\, f_{20} \end{cases} .$$

Repeating this kind of reasoning (we omit the calculations since they are purely mechanical) one obtains:

$$R_{11} : y_0\, y_1 = x_0 z_2$$

$$R_{12} : y_0\, y_2 = x_1 z_2$$

$$R_{15} : y_0\, z_1 = x_0\, (\delta\, y_2^2 + x_2^2\, g_1 + \lambda\, f_{20})$$

$$R_{16} : y_0\, z_2 = x_1\, (\delta\, y_2^2 + x_2^2\, g_1) + x_0 f_{20}$$

$$R_{19} : z_1 z_2 = \delta\, y_1 y_2^2 + x_2^2 y_1 g_1 + \lambda\, y_1\, f_{20} + \varepsilon_3\, x_2^4\, y_0 + x_1\, q_{19} + x_0\, f_{19}$$

$$R_{20} : z_2^2 = \delta\, y_2^3 + x_2^2 y_2 g_1 + y_1\, f_{20} + \alpha\, x_2^4 y_0 + x_1 q_{20} + x_0 f_{20}$$

and $f_{20} = \alpha_0\, y_2^2 + \alpha_1\, x_2^2\, y_2$.

In the same way it is possible to obtain some information about the coefficients of the other polynomials but I have not done the calculations.

(5.3) **Proposition** If $F = 2C$ is a double fibre of genus 3, with C 2-connected, $R(F, K_F)$ can be presented as $k[X_0, X_1, X_2, Y_0, Y_1, Y_2, Z_1, Z_2] / I$ where I is the ideal generated by the 20 polynomials in table V-1, with $\delta, \lambda \neq 0$.

Proof Let φ_n be as in (II.6.3) the restriction maps $\varphi_n : R_n \to H^0(C, nK_F)$. We have seen that $\operatorname{Ker}\varphi_n$ could be identified with $H^0(C, (n-1)\, K_F + R_C)$ and we are going to consider the following filtration of R_n:

$0 \hookrightarrow \operatorname{Ker}\varphi_n \hookrightarrow R_n$.

Remark that $\dim \operatorname{Ker}\varphi_n = \begin{cases} 2 & \text{for } n = 1 \\ 2n-1 & \text{for } n \geq 2 \end{cases}$ and also that as in (5.2)

for any two elements $\mu \in \operatorname{Ker}\varphi_n$, $v \in \operatorname{Ker}\varphi_m$, $\mu v = 0$.

Let us recall (II.6.2) that $\mathcal{O}_C(K_F) \not\cong \mathcal{O}_C(K_C)$ and $\mathcal{O}_C(nK_F) \cong \mathcal{O}_C(n\, K_C)$, for n even.

Let (s_0, s_1) be any basis of $H^0(C, K_C)$. Then by (I.9.4) $(s_0^2, s_0 s_1, s_1^2)$ is a basis of $H^0(C, 2K_C)$. If $\varepsilon \in \operatorname{Im}\varphi_1$, then $\varepsilon^2 \in H^0(C, 2K_C)$ and thus $\varepsilon^2 = \alpha_0 s_0^2 + \alpha_1\, s_0 s_1 + \alpha_2 s_1^2$. Since $\mathcal{O}_C(K_F) \not\cong \mathcal{O}_C(K_C)$, for ε nonzero we cannot have simultaneously $\alpha_1 = \alpha_2 = 0$ or $\alpha_0 = \alpha_1 = 0$.

Let then (x_0, x_1, x_2) be a basis of R_1 such that $x_0, x_1 \in \operatorname{Ker} \varphi_1$. Then $x_{2|C}$ generates $H^0(C, K_F)$. We will assume that x_0, x_1 were chosen in such a way that in the identification of $\operatorname{Ker} \varphi_1$ with $H^0(C, K_C)$ x_0, x_1 correspond to s_0, s_1 such that $x_2^2 = s_0^2 - \lambda s_0 s_1$, with $\lambda \neq 0$.

In degree 2 we have $x_0^2 = x_0 x_1 = x_1^2 = 0$ and $x_0 x_2, x_1 x_2$ are independent elements of $\operatorname{Ker} \varphi_2$. Because C is 2-connected it is easy to see that $\mathcal{O}_C(K_F + K_C)$ is generated by its global sections. Then when identifying $\operatorname{Ker} \varphi_2$ with $H^0(C, K_F + K_C)$ the new generator $y_0 \in \operatorname{Ker} \varphi_2$ corresponds to $d_0 \in H^0(C, K_F + K_C)$ which has no common zeros with x_2. We can choose the two new generators y_1, y_2 for $R_2 \backslash \operatorname{Ker} \varphi_2$, such that $y_{1|C} = \delta s_0 s_1$, $y_{2|C} = \delta s_1^2$, with $\delta \neq 0$. Then y_2 has no common zeros with x_2^2.

Now in degree 3, since $y_0 \in \operatorname{Ker} \varphi_2$ we have

$$x_0 y_0 = 0$$
$$x_1 y_0 = 0 .$$

By (II.6.3) the subspace of $\operatorname{Ker} \varphi_3$ generated by $(x_0 x_2^2, x_1 x_2^2, x_0 y_1, x_0 y_2, x_1 y_1, x_1 y_2)$ has dimension 4. With the identifications made above, we have in R

$$x_0 y_1 = \delta x_1 x_2^2 + \lambda x_0 y_2$$
$$x_1 y_1 = x_0 y_2 .$$

Using the free pencil trick we have that

$$\operatorname{Im} \{x_0 x_2 H^0(C, 2K_F) + y_0 H^0(C, 2K_F) \longrightarrow \operatorname{Ker} \varphi_n\}$$

has dimension 6. Thus $(x_0x_2^3, x_0x_2y_1, x_0x_2y_2, x_2^2y_0)$ are independent in R_4 and so $(x_0x_2^2, x_0y_1, x_0y_2, x_2y_0)$ are independent in R_3. By the choices made for x_1, y_2, y_0, $\langle x_1y_2\rangle \cap \langle x_0x_2^2, x_0y_1, x_0y_2, x_2y_0\rangle = \{0\}$ and so we get a basis for $\operatorname{Ker}\varphi_3$.

We also have (x_2^3, x_2y_1, x_2y_2) independent. Since $\operatorname{Im}\varphi_3$ is 5-dimensional there will be two new generators z_1, z_2 for R_3.

The map $H^0(C, K_F + K_C) \otimes H^0(C, K_C) \longrightarrow H^0(C, 3K_F)$ is surjective and thus we can assume that $z_{1|C} = \delta s_0 d_0$, $z_{2|C} = \delta s_1 d_0$.

In degree 4, by Castelnuovo's lemma, $\operatorname{Ker}\varphi_1 \otimes H^0(C, 3K_F) \to \operatorname{Ker}\varphi_4$ is surjective and because of the choices made before we have

$$x_1z_1 = x_0z_2.$$

Hence $(x_0x_2^3, x_1x_2^3, x_0x_2y_2, x_1x_2y_2, x_0z_2, x_1z_2, x_0z_1)$

form a basis of $\operatorname{Ker}\varphi_4$.

Since $y_0 \in \operatorname{Ker}\varphi_2$

$$y_0^2 = 0$$

and with the identifications made before we also have in R

$$y_0y_1 = x_0z_2$$
$$y_0y_2 = x_1z_2$$

$$x_0(z_1 - \lambda z_2) = \delta x_2^2 y_0$$

and $\qquad y_{1|C}^2 = (\delta x_2^2 y_2 + \lambda y_1 y_2)_{|C}$,

hence $\qquad y_1^2 = \delta\, x_2^2\, y_2 + \lambda\, y_1 y_2 + \varepsilon_0\, x_2^2\, y_0 + x_0\, p + x_1\, q$

with $p, q \in k[x_2, y_2, z_2]$.

Now let us observe that the map

$$s_0^2.H^0(C, K_F+K_C) + s_1^2.H^0(C, K_F+K_C) \to H^0(C, 2K_F+2K_C) \cong H^0(C, 4K_F)$$

is surjective and thus

$$d_0^2 = P_4(s_0,s_1) + d_0\, P_2(s_0, s_1)\,.$$

By changing d_0 we can assume that $P_2 = 0$. Remark then that the coefficient of s_1^2 in P_4 is different from zero, and thus this relation can be written

$$\delta\, d_0^2 = \delta^2 a s_1^4 + \delta^2 b s_1^3 s_0 + \delta c(s_0^2 - \lambda\, s_0 s_1)s_1^2 + \delta\, d(s - \lambda\, s_0 s_1)s_0 s_1 + e(s_0^2 - \lambda\, s_0 s_1)^2$$

with $a \neq 0$.

Using this relation and the identifications made before we have in degree 5 , relations

$$y_1 z_2 = y_2 z_1 + p_1\,, \quad \text{with } p_1 \in \operatorname{Ker} \varphi_5$$

$$y_1(z_1 - \lambda\, z_2) = \delta\, x_2^2\, z_2 + p_2\,, \quad \text{with } p_2 \in \operatorname{Ker} \varphi_5$$

$$y_0\, z_1 = x_0\, P$$

$$y_0\, z_2 = x_1\, P$$

and in degree 6

$$z_1(z_1 - \lambda\, z_2) = \delta\, x_2^2\, P + p_3$$

$$z_1\, z_2 = y_1\, P + p_4$$

$$z_2^2 = y_2 P + p_5$$

where $P = a\, y_2^2 + b\, y_1 y_2 + c\, x_2^2\, y_2 + d\, x_2^2\, y_1 + e\, x_2^4$

and p_3 , p_4, $p_5 \in \operatorname{Ker} \varphi_6$.

The coefficients of the polynomials $p_1, \ldots, p_5$ are also related but I have not done that calculation.

Section 6. The canonical ring of a 2-connected fibre.

(6.1) Theorem If F is a 2-connected genus 3 fibre, $R(F, K_F)$ can be presented as

$$R(F, K_F) = k[X_0, X_1, X_2, Y_1] / (F_2, F_4)$$

where

$$F_2 : X_2^2 - \lambda Y_1 - P_2(X_0, X_1) - X_2 P_1(X_0, X_1)$$

$$F_4 : Y_1^2 - Q_4(X_0, X_1) - X_2 Q_3(X_0, X_1) - X_2^2 Q_2(X_0, X_1) - X_2^3 Q_1(X_0, X_1).$$

F is hyperelliptic if $\lambda = 0$ and F is not hyperelliptic if $\lambda \neq 0$.

Proof By (I.9.10), $R(F, K_F)$ is generated by its elements of degree lesser than 2 and by (I.6.3), K_F is generated by its global sections.

Let (x_0, x_1, x_2) be a basis of R_1 such that (x_0, x_1) have no common zeros. By the free pencil trick $x_0 R_1 + x_1 R_1$ has codimension 1 in R_2 and $x_0 R_3 + x_1 R_3 = R_4$.

Now in degree 2 either $x_2^2 \in x_0R_1 + x_1R_1$ or not. If not; we have $R_2 = (x_0^2, x_0x_1, x_1^2, x_0x_2, x_1x_2, x_2^2)$. Otherwise we have $x_2^2 = P_2(x_0, x_1) ++x_2 P_1(x_0, x_1)$ in R and we will need a new generator y_1.

Since $x_0 R_3 + x_1 R_3 = R_4$ we will have $y_1^2 = P_4(x_0, x_1) + x_2 P_3(x_0, x_1) + y_1 P_2(x_0, x_1) + x_2y_1 P_1(x_0, x_1)$. Changing y_1 we can assume that the right-hand side has no terms in y_1.

If $R_1.R_1 = R_2$

$$x_2^4 = P_4 (x_0, x_1) + x_2 P_3(x_0, x_1) + x_2^2 P_2(x_0, x_1) + x_2^3 P_1(x_0, x_1).$$

Now by a simple dimension count and using the fact that $R_1.R_n = R_{n+1}$ for $n \geq 3$, it is easy to check that there are not any other independent relations in R. Thus we have the result.

Section 7. A deformation calculation

(7.1) For all theory of deformations refer to [A] and [R-2].

For non 2-connected genus 3 fibres, the ideal of relations and number of syzygies between these relations are very big. Because of this explicit computations of deformations are very hard to do.

Nevertheless, for each ring R of Type III (II,I,resp.) I have written down flat families $\{R_t\}$ with $R_0=R$ and R_t of Type II (I, 2-connected,resp.) for $t\neq 0$. For generic Type I,II,III rings R I have also written down flat families $\{R_t\}$ with $R_0=R$ and R_t an irreducible genus 3 curve,for $t\neq 0$.

I do not present these calculations here because they were done with different presentations of R.

The main point of this section is to give an essentially complete treatment of deformations for Type I fibres containing an elliptic tail appearing with multiplicity one in F (as in 4.1(i)). In this case a calculation deformation can easily be carried out due to the specific format of the relations.In fact we have:

(7.2) **Proposition** If F is as in (4.1.(i)) then:

(i) a set of generators for the ideal I is given by the 2×2 minors of the matrix

$$A = \begin{pmatrix} 0 & X_2 & X_1 & Y_1 \\ X_0 & Y_2 & B & Z \end{pmatrix}$$

and

the entries of the symmetric matrix $A\,M\,{}^tA$

$$\text{where} \quad M = \begin{bmatrix} h_4 & 0 & 0 & 0 \\ 0 & h_3 & h_2 & 0 \\ 0 & h_2 & h_1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

with $h_1, h_2, h_3, h_4 \in k[X_0, X_1, X_2, Y_1, Y_2]$.

(ii) All the syzygies between elements of I can all be deduced from this format. There are two groups of syzygies - one obtained by doubling a line of A and taking the 3×3 minors of the matrix thus obtained and the other obtained from the identity

$$(A^*A)\,(M\,A^T) = A^*\,(A\,M\,A^T)$$

$$\text{where } A^* = \begin{bmatrix} -X_0 & 0 \\ -Y_2 & X_2 \\ -B & X_1 \\ -Z & Y_1 \end{bmatrix}.$$

(iii) Every first order (flat) deformation $\bar{R}$ can be obtained as a quotient of $k[X_0, X_1, X_2, Y_1, Y_2, Z]\,[t]\,/\,(t^2)$ by an ideal $\tilde{I}$ generated by the 2×2 minors of

$$\tilde{A} = \begin{pmatrix} t\alpha & X_2 & X_1 & Y_1 \\ X_0 & Y_2 & B+t\Gamma & Z \end{pmatrix}$$

and the entries of

$$\tilde{A}\,\tilde{M}\,{}^t\tilde{A}$$

where

$$\tilde{M} = \begin{bmatrix} h_4 - tg_4 & 0 & 0 & 0 \\ 0 & h_3 - tg_3 & h_2 & -tg_2 \\ 0 & h_2 & h_1 - tg_1 & 0 \\ 0 & -tg_2 & 0 & -1 \end{bmatrix}.$$

and $\alpha \in \mathbb{C}$, $\Gamma = \gamma_0 X_2^2 + \gamma_1 Y_1 + \gamma_2 Y_2$, $g_1, g_2, g_3, g_4 \in k[X_0, X_1, X_2, Y_1, Y_2]$.

Remarks (1) This format and its properties are due to Duncan Dicks [D] and Miles Reid [R-2].

(2) Proposition (7.2) is proved below.

(7.3) Remark As a consequence of (7.2(ii)) we see that the format of the relations in R is flexible in the sense of [R-2], i.e. small variations of the entries of A and M lead to flat deformations of R.

As a consequence of (7.2(iii)) we have that all flat deformations of R are in fact obtained in this way (cf. [R-2] section 5)

(7.5) Remark Looking at the way in which first order deformations can be written it is easy to see that F can be obtained as the degenerate fibre of a family of hyperelliptic genus 3 fibres if and only if B does not involve Y_1, i.e. $\lambda\gamma=0$ (compare 4.1(i)).

(7.5) Remark For rings of Type I corresponding to non-reduced fibres we can present I as the ideal generated by the 2x2 minors of a matrix A plus the entries of A L where L is a 4x2 matrix. In this case the syzigies also come from doubling the rows of A and the identity $(A^*A)L=A^*(AL)$. Due to the nilpotent structure of R it does not seem possible to decompose L as a product of a symmetric matrix M by A^T. A similar difficulty appears in [D] (15.2).

Proof of (7.2) For details and explanations of the proof of (iii) refer to [R-2] section 5, where a very similar calculation is done.

(i)We have seen that in case (i) of (4.1) $R = k[X_0, X_1, X_2, Y_1, Y_2, Z] / I$ where I is generated by

$$R_1 = X_0X_2$$
$$R_2 = X_0X_1$$
$$R_3 = X_0Y_1$$
$$R_4 = X_2B-X_1Y_2$$
$$R_5 = X_2Z-Y_1Y_2$$
$$R_6 = X_1Z-Y_1B$$

$$U = Y_1^2-\alpha_0X_1^2Y_1-\alpha_1X_1X_2Y_1-\alpha_2X_2^2Y_1-\gamma_0X_1X_2^3-\gamma_1X_1^2X_2^2-\gamma_2X_1^3X_2-$$
$$-\gamma_3X_1^4 - \varepsilon X_2^4-X_2^2Y_2$$

$$V = Y_1Z-\alpha_0X_1^2Z-\alpha_1X_1X_2Z-\alpha_2X_2^2Z-\gamma_0X_2^3B-\gamma_1X_2^2X_1B-\gamma_2X_2X_1^2B-$$
$$-\gamma_3X_1^3B - \varepsilon X_2^3Y_2-X_2Y_2^2$$

and

$$W = Z^2-\alpha_0X_1BZ-\alpha_1X_2BZ-\alpha_2X_2Y_2Z - (\lambda X_2Y_2 + \lambda\gamma Z) -$$
$$- (\gamma_0 X_2^3 + \gamma_1 X_1X_2^2 \ \gamma_2 X_2X_1^2 + \gamma_3 X_1^3) - \varepsilon X_2^2Y_2^2-Y_2^3 - \varepsilon_0 X_0^6 - \varepsilon_1 X_0^4Y_2.$$

Using the other relations we can rewrite U , V , W as

$$\bar{U} = Y_1^2 - X_1^2 h_1 - 2X_1X_2h_2 - X_2^2h_3 - 0\,h_4$$

$$\bar{V} = Y_1Z - X_1Bh_1 - (X_2B + X_1Y_2)h_2 - X_2Y_2h_3 - 0\,h_4$$

$$\bar{W} = Z^2 - B^2h_1 - 2\,Y_2\,B\,h_2 - Y_2^2h_3 - X_0^2\,h_4$$

with

$$h_1 = \alpha_0 Y_1 + \gamma_1 X_2^2 + \gamma_2 X_1 X_2 + \gamma_3 X_1^2$$

$$h_2 = \frac{1}{2}\alpha_1\,Y_1$$

$$h_3 = \alpha_2 Y_1 + \gamma_0 X_1 X_2 + \varepsilon\,X_2^2 + Y_2$$

$$h_4 = \varepsilon_0\,X_0^4 + \varepsilon_1\,X_0^2 Y_2\,.$$

Let $\quad A = \begin{bmatrix} 0 & X_2 & X_1 & Y_1 \\ X_0 & Y_2 & B & Z \end{bmatrix}$

and

$$M = \begin{bmatrix} h_4 & 0 & 0 & 0 \\ 0 & h_3 & h_2 & 0 \\ 0 & h_2 & h_1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Then $R_1, \ldots, R_6$ are the 2×2 minors of A and $A\,M\,A^T = \begin{bmatrix} -\bar{U} & -\bar{V} \\ -\bar{V} & -\bar{W} \end{bmatrix}$

(ii) It is an easy exercise to check that all the syzygies holding between these polynomials can be obtained from the above presentation in this way.Remark that

$$M\,A^T = \begin{bmatrix} 0 & X_0h_4 \\ X_2h_3+X_1h_2 & Y_2h_3+Bh_2 \\ X_2h_2+X_1h_1 & Y_2h_2+Bh_1 \\ -Y_1 & -Z \end{bmatrix}$$

and

$$A^*A = \begin{bmatrix} 0 & -R_1 & -R_2 & -R_3 \\ R_1 & 0 & R_4 & R_5 \\ R_2 & -R_4 & 0 & R_6 \\ R_3 & -R_5 & -R_6 & 0 \end{bmatrix}$$

Doubling lines in A we get syzygies

$$\begin{aligned}
&\zeta_1 : X_1R_1 - X_2R_2 \\
&\zeta_2 : X_2R_3 - Y_1R_1 \\
&\zeta_3 : X_1R_3 - Y_1R_2 \\
&\zeta_4 : X_2R_6 - X_1R_5 + Y_1R_4 \\
&\zeta_5 : X_0R_4 + Y_2R_2 - BR_1 \\
&\zeta_6 : X_0R_5 - Z\,R_1 + Y_2R_3 \\
&\zeta_7 : X_0R_6 - Z\,R_2 + BR_3 \\
&\zeta_8 : Y_2R_6 - BR_5 + ZR_4 \ .
\end{aligned}$$

From $(A^*A)\,(M\,A^T) = A^*\,(A\,M\,A^T)$ we get syzygies

$$\sigma_1 : X_0\dot{U} - Y_1R_3 + (X_1h_1 + X_2h_2)\, R_2 + (X_2h_3 + X_1h_2)\, R_1$$
$$\sigma_2 : X_0\bar{V} - Z\,R_3 + (Y_2h_2 + Bh_1)\, R_2 + (Y_2h_3 + Bh_2)\, R_1$$
$$\sigma_3 : Y_2U - X_2\bar{V} + Y_1R_5 - (X_2h_2 + X_1h_1)\, R_4$$
$$\sigma_4 : Y_2\bar{V} - X_2\bar{W} + ZR_5 - (Y_2h_2 + Bh_1)\, R_4 - X_0h_4\, R_1$$
$$\sigma_5 : BU - X_1\bar{V} + Y_1R_6 + (X_2h_3 + X_1h_2)\, R_4$$
$$\sigma_6 : B\bar{V} - X_1\bar{W} + ZR_6 + (Y_2h_3 + Bh_2)\, R_4 - X_0h_4R_2$$
$$\sigma_7 : ZU - Y_1\bar{V} + (X_2h_3 + X_1h_2)R_5 + (X_2h_2 + X_1h_1)\, R_6$$
$$\sigma_8 : Z\bar{V} - Y_1\bar{W} + (Y_2h_3 + Bh_2)\, R_5 + (Y_2h_2 + Bh_1)\, R_6 - X_0h_4\, R_3$$

(iii) To prove (iii) we need to calculate all first order deformations. We will make some coordinate changes along the way to simplify the calculations.

Let
$$\tilde{R}_1 = X_0X_2 + tR'_1$$
$$\tilde{R}_2 = X_0X_1 + tR'_2$$
$$\tilde{R}_3 = X_0Y_1 + tR'_3$$
$$\tilde{R}_4 = X_2B - X_1Y_2 + tR'_4$$
$$\tilde{R}_5 = X_2Z - Y_1Y_2 + tR'_5$$
$$\tilde{R}_6 = X_1Z - Y_1B + tR'_6\ .$$

Changing X_0, X_1, X_2 we can assume that

$$R'_1 = \varepsilon_0 X_1^2 + \varepsilon_1 X_1X_2 + \varepsilon_2 X_2^2 + \varepsilon_3 Y_1 + \varepsilon_4 Y_2$$

$$R'_2 = \gamma_0 X_2^2 + \gamma_1 Y_1 + \gamma_2 Y_2\ .$$

Changing Y_1 we can assume that

$$R'_3 = \mu_0 X_1^3 + \mu_1 X_1^2 X_2 + \mu_2 X_2^2 + \mu_3 X_2^3 + \mu_4 X_1 Y_1 + \mu_5 X_2 Y_1 + \mu_7 X_2 Y_2 + \mu_8 Z.$$

Changing Y_2 we can assume that

$$R'_4 = \gamma'_0 X_0^3 + \gamma'_1 X_0 Y_2 + \gamma'_2 X_2^3 + \gamma'_2 X_2 Y_2 + \gamma'_4 Z + \gamma'_5 X_2 Y_1 .$$

Changing Z we can assume that

$$R'_5 = a_0 X_0^4 + a_1 X_0^2 Y_2 + a_2 X_0 Z + a_3 X_1^4 + a_4 X_1^2 Y_1 + a_5 Y_2^2$$

and we have

$$R'_6 = \beta_0 X_0^4 + \beta_1 X_0^2 Y_2 + \beta_2 X_0 Z + \beta_3 X_1^4 + \ldots + \beta_7 X_2^4 + \beta_8 X_1^2 Y_1 + ..$$
$$+ \beta_{10} X_2^2 Y_1 + \beta_{11} X_2^2 Y_2 + \beta_{12} X_2 Z + \beta_{13} Y_2^2 .$$

In order for ζ_1 to extend to a syzygy in $\bar{R}$ we need to have $X_1 R'_2 - X_2 R'_1 \in I$ and so using the generators of I we get that

$\varepsilon_0 X_1^3 + \varepsilon_1 X_1^2 X_2 + \varepsilon_2 X_2^2 X_1 + \varepsilon_3 X_1 Y_1 - (\gamma_0 - \varepsilon_4 \lambda) X_2^3 - (\gamma_1 - \varepsilon_4 \gamma\lambda) X_2 Y_1 - \gamma_2 X_2 Y_2$ is in I.

But all the elements appearing above are part of a complementary basis for I_3 in $k[X_0, X_1, X_2, Y_1, Y_2, Z]_3$ and thus we have

$$\begin{cases} \varepsilon_0 = \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \gamma_2 = 0 \\ \gamma_0 = \varepsilon_4 \lambda \\ \gamma_1 = \varepsilon_4 \lambda \gamma \end{cases}$$

and so $R'_1 = \alpha' Y_2, R'_2 = \alpha' B$.

In the same way, from ζ_2 we get $R'_3 = \alpha' Z$. Now from ζ_5 we get $\gamma'_0 = \gamma'_1 = \gamma'_4 = 0$, and from ζ_6 we get $\alpha_0 = a_1 = a_2 = a_5 = 0$ .From ζ_7 we get $\beta_0 = \beta_1 = \beta_2 = \beta_{13} = 0$.

From ζ_4 we get

$$\begin{cases} \alpha_3 = 0 \\ \alpha_4 = 0 \\ \beta_3 X_1^4 + \beta_4 X_1^3 X_2 + \beta_5 X_1^2 X_2^2 = -\gamma'_5 X_1^2 P_1 \\ \beta_6 = -\gamma_0 \gamma'_5 \\ \beta_7 X_2^2 + \beta_{11} Y_2 = -\gamma'_5 Q \\ \beta_8 = -\gamma'_5 \alpha_0 \\ \beta_9 = -\gamma'_5 \alpha_1 \\ \beta_{10} = -\gamma'_5 \alpha_2 - \gamma'_2 \\ \beta_{12} = -\gamma'_3 \end{cases}$$

giving $R'_4 = \gamma'_2 X_2^3 + \gamma'_3 X_2 Y_2 + \gamma'_5 X_2 Y_1 = X_2 \Gamma$

$R'_5 = 0$

$R'_6 = -\gamma'_2 X_2^2 Y_1 - \gamma'_3 X_2 Z - \gamma'_5 (X_1^2 h_1 + 2 X_1 X_2 h_2 + X_2^2 h_3)$

Remark that, using the other relations, we can write $R'_6 = -Y_1 \Gamma$ .Now it is easy to check that ζ_8 is automatically satisfied.

Write now

$$U' = \mu_0 X_0^4 + \mu_1 X_0^2 Y_2 + \mu_2 X_0 Z + \mu_3 Y_2^2 + \mu_4 X_1^4 + \mu_5 X_1^3 X_2 + \mu_6 X_1^2 X_2^2$$

$$+ \mu_7 X_1 X_2^3 + \mu_8 X_2^4 + \mu_9 X_1^2 Y_1 + \mu_{10} X_1 X_2 Y_1 + \mu_{11} X_2^2 Y_1 + \mu_{12} X_2^2 Y_2 + \mu_{13} X_2 Z$$

$$V' = \nu_0 X_0^5 + \nu_1 X_0^3 Y_2 + \nu_2 X_0^2 Z + \nu_3 X_0 Y_2^2 + \nu_4 Y_2 Z + \nu_5 X_1^5 + \dots +$$

$$+ \nu_{10} X_2^5 + \nu_{11} X_1^3 Y_1 + \dots + \nu_{14} X_2^3 Y_1 + \nu_{15} X_2^3 Z + \nu_{16} X_2^2 Z + \nu_{17} X_2 Y_2^2$$

and

$$W' = \delta_0 X_0^6 + \delta_1 X_0^4 Y_2 + \delta_2 X_0^3 Z + \delta_3 X_0^2 Y_2^2 + \delta_4 X_0 Y_2 Z + \delta_5 Y_2^3 +$$

$$+ \delta_6 X_1^6 + \ldots + \delta_{12} X_2^6 + \delta_{13} X_1^4 Y_1 + \ldots + \delta_{17} X_2^4 Y_1 + \delta_{18} X_2^4 Y_2 +$$

$$+ \delta_{19} X_2^3 Z + \delta_{20} X_2^2 Y_2^2 + \delta_{21} X_2 Y_2 Z \ .$$

From σ_1 we obtain $\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0$ and from σ_2

$$\nu_0 X_0^5 + \nu_1 X_0^3 Y_2 + \nu_2 X_0^2 Z + \nu_3 X_0 Y_2^2 + \nu_4 Y_2 Z = \alpha' X_0 h_4 \ .$$

From σ_3 we obtain

$$\begin{cases} \mu_{13} = 0 \\ V' = \alpha' X_0 h_4 + \mu_4 X_1^3 B + \mu_5 X_1^2 X_2 B + \mu_6 X_1 X_2^2 B + \mu_7 X_2^3 B + \\ + \mu_8 X_2^3 Y_2 + \mu_9 X_1 Y_1 B + \mu_{10} X_2 Y_1 B + \mu_{11} X_2^2 Z + \mu_{12} X_2 Y_2^2 \\ - (X_2 h_2 + X_1 h_1) \Gamma' \end{cases}$$

and from σ_4

$$W' = \delta_0 X_0^6 + \delta_1 X_0^4 Y_2 + \delta_2 X_0^3 Z + \delta_3 X_0^2 Y_2^2 + \delta_4 X_0 Y_2 Z +$$

$$+ \mu_4 X_1^2 B^2 + \mu_5 X_1 X_2 B^2 + \mu_6 X_2^2 B^2 + \mu_7 X_2^2 Y_2 B + \mu_8 X_2^2 Y_2^2$$

$$+ \mu_9 Y_1 B^2 + \mu_{10} Y_1 Y_2 B + \mu_{11} X_2 Y_2 Z + \mu_{12} Y_2^3 - 2(Y_2 h_2 + B h_1)\Gamma \ .$$

Now $\sigma_5, \sigma_6, \sigma_7, \sigma_8$ are satisfied automatically.

We can again change Z so that $\delta_2 = \delta_4 = 0$. Notice that this does not change the form of the other polynomials. We can then write

$$U' = X_1^2\, g_1 + 2\, X_2 Y_1 g_2 + X_2^2\, g_3$$

$$V' = X_1 B\, g_1 + (Y_1 Y_2 + X_2 Z)\, g_2 + X_2 Y_2\, g_3 - (X_2 h_2 + X_1 h_1)\, \Gamma - \alpha' X_0 h_4$$

$$W' = B^2\, g_1 + 2\, Y_2 Z\, g_2 + Y_2^2\, g_3 - 2\, (Y_2 h_2 + B h_1)\, \Gamma + X_0^2 Y_4$$

where

$$g_1 = \mu_4\, X_1^2 + \mu_5\, X_1 X_2 + \mu_6\, X_2^2 + \mu_9\, Y_1$$

$$g_2 = \frac{1}{2}(\mu_{10}\, X_1 + \mu_{11}\, X_2)$$

$$g_3 = \mu_7\, X_1 X_2^2 + \mu_8\, X_2^2 + \mu_{12}\, Y_2$$

$$g_4 = \delta_0\, X_0^4 + \delta_1\, X_0^2 Y_2 + \delta_2\, Y_2^2$$

and then we have $\tilde{R}_1, \ldots \tilde{R}_6$ given by the 2x2 minors of

$$\tilde{A} = \begin{pmatrix} t\alpha & X_2 & X_1 & Y_1 \\ X_0 & Y_2 & B + t\Gamma & Z \end{pmatrix}$$

(with $\Gamma = \gamma'_0 X_2^2 + \gamma'_1 Y_1 + \gamma'_2 Y_2$) and the entries of $\tilde{A}\,\tilde{M}\,\tilde{A}^T$ where

$$\tilde{M} = \begin{bmatrix} h_4 - tg_4 & 0 & 0 & 0 \\ 0 & h_3 - tg_3 & h_2 & -tg_2 \\ 0 & h_2 & h_1 - tg_1 & 0 \\ 0 & -tg_2 & 0 & -1 \end{bmatrix}.$$

End of proof of (7.2).

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